ASYMPTOTIC DISTRIBUTION OF FIXED POINTS OF PATTERN-AVOIDING INVOLUTIONS

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1. Abstract of talk

For a variety of pattern-avoiding classes, we describe the limiting distribution for the number of fixed points for involutions chosen uniformly at random from that class. In particular we consider monotone patterns of arbitrary length as well as all patterns of length 3. For monotone patterns we utilize the connection with standard Young tableaux with at most k rows and involutions avoiding a monotone pattern of length k [3, 4, 2]. For every pattern of length 3 we give the bivariate generating function with respect to fixed points for the involutions that avoid that pattern, and where applicable apply tools from analytic combinatorics [1] to extract information about the limiting distribution from the generating function. Many well-known distributions appear.

Let $fp(\pi)$ denote the number of number of fixed points of π. For a class of permutations $\mathcal{C}$ we define the bivariate generating function with respect to fixed points as

$$F_{\mathcal{C}}(x,t) = \sum_{\pi \in \mathcal{C}} x^{fp(\pi)} t^{|\pi|}.$$

For a pattern τ we let $\mathbf{Iv}(\tau)$ denote the class of permutations that avoid τ. The generating functions for patterns in S_3 are as follows:

- For $\mathbf{Iv}(231) = \mathbf{Iv}(312)$ we have

$$F_{\mathbf{Iv}_n(213)}(x,t) = \sum_{\sigma \in \mathbf{Iv}(231)} x^{fp(\sigma)} t^{|\sigma|} = \frac{1-t^2}{1-2t^2-xt}.$$

- For each $\tau \in \{321, 132, 213\}$ we have

$$F_{\mathbf{Iv}(\tau)}(x,t) = \frac{2}{1-2xt+\sqrt{1-4t^2}}.$$

- And finally for for $\mathbf{Iv}(123)$ we have

$$F_{\mathbf{Iv}(123)}(x,t) = \sum_{\rho \in \mathbf{Iv}123} x^{fp(\rho)} t^{|\rho|} = 1 + (tx + t^2(1+x^2)) \left(\frac{1-\sqrt{1-4t^2}}{2t^2\sqrt{1-4t^2}} \right).$$

In some cases, we are able to use straightforward generating function arguments coupled with classical analytic combinatorics to compute the asymptotic distribution of fixed points. Using the above generating function for $\mathbf{Iv}(231)$ we have the following result.

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Theorem 1. *Fix $\tau \in \{231, 312\}$. If Π_n is a uniformly random element of $\boldsymbol{Iv}_n(\tau)$ then $\mathbb{E}(fp(\Pi_n)) = \frac{n}{3} + O(1)$, $Var(fp(\Pi_n)) = \frac{8}{27}n + O(1)$ and*

$$\frac{fp(\Pi_n) - \frac{1}{3}n}{\sqrt{8n/27}} \to_d Z,$$

where Z is a standard normal random variable.

Some cases, however, cannot be easily done using off-the-shelf results from analytic combinatorics and require different methods. Using the analysis of Young tableaux in [2] we have may understand the asymptotic distribution of fixed points for involutions avoiding long monotone sequences.

Theorem 2. *Fix $k \in \{2, 3, \dots\}$ and let Π_n be a uniformly random element of $\boldsymbol{Iv}_n((k+1)k \cdots 321)$. Let M be a random matrix drawn from the $k \times k$ Gaussian Orthogonal Ensemble conditioned to have trace 0 and let $\Lambda_1 \geq \cdots \geq \Lambda_k$ be the ranked eigenvalues of M.*

(a) If k is even then

$$\sqrt{\frac{k}{n}} fp(\Pi_n) \to_d \sum_{j=1}^{k} (-1)^{j+1} \Lambda_j.$$

(b) If k is odd then

$$\sqrt{\frac{k}{n}} \left(fp(\Pi_n) - \frac{n}{k} \right) \to_d \sum_{j=1}^{k} (-1)^{j+1} \Lambda_j.$$

Theorem 3. *If Π_n is a uniformly random element in $\boldsymbol{Iv}_n(123 \cdots k(k+1))$ then*

$$fp(\Pi_{2n}) \to_d X_{even}$$

and

$$fp(\Pi_{2n-1}) \to_d X_{odd}\,,$$

where X_{even} has density function given by

$$\mathbb{P}(X_{even} = i) = \begin{cases} \frac{\binom{k}{i}}{2^{k-1}} & i \text{ is even}\,, \\ 0 & i \text{ is odd}\,, \end{cases}$$

and X_{odd} has density function given by

$$\mathbb{P}(X_{odd} = i) = \begin{cases} \frac{\binom{k}{i}}{2^{k-1}} & i \text{ is odd}\,, \\ 0 & i \text{ is even}\,. \end{cases}$$

References

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