1. The block number of a permutation

1.1. Definitions. Direct sums and the block decomposition of permutations appear naturally in the study of pattern-avoiding classes [3, 4]. Let \( \pi \in S_m \) and \( \sigma \in S_n \). The direct sum of \( \pi \) and \( \sigma \) is the permutation \( \pi \oplus \sigma \in S_{m+n} \) defined by

\[
\pi \oplus \sigma := \begin{cases} 
\pi(i), & \text{if } i \leq m; \\
\sigma(i - m) + m, & \text{otherwise}.
\end{cases}
\]

For example, if \( \pi = 312 \) and \( \sigma = 2413 \) then \( \pi \oplus \sigma = 3125746 \); see Figure 1.

![Figure 1](image)

**Figure 1.** The permutation \( 312 \oplus 2413 = 3125746 \)

A nonempty permutation which is not the direct sum of two nonempty permutations is called \( \oplus \)-irreducible. Each permutation \( \pi \) can be written uniquely as a direct sum of \( \oplus \)-irreducible ones, called the blocks of \( \pi \); their number, denoted by \( bl(\pi) \), is the block number of \( \pi \). Equivalently,

\[
bl(\pi) = |\{1 \leq i \leq n : (\forall j \leq i) \, \pi(j) \leq i\}|.
\]

1.2. Counting 321-avoiding permutations by block number. Recall the \( n \)-th Catalan number, \( C_n := \frac{1}{n+1} \binom{2n}{n} \), and its generating function \( c(x) := \sum_{n \geq 0} C_n x^n \). For each \( 0 \leq k \leq n \), the \( n \)-th \( k \)-fold Catalan number \( C_{n,k} \) is the coefficient of \( x^n \) in \((xe(x))^k\). These numbers are also called ballot numbers, and form the Catalan triangle [15, A009766].

A permutation \( \pi \in S_n \) is 321-avoiding if the sequence \((\pi(1), \ldots, \pi(n))\) contains no decreasing subsequence of length 3. Denote by \( S_n(321) \) the set of 321-avoiding permutations in \( S_n \).
Proposition 1.1. [1] For any fixed positive integer $k$, the ordinary generating function for the number of $321$-avoiding permutations in $S_n$ with exactly $k$ blocks is $(xc(x))^k$.

Recall the descent set of a permutation $\pi \in S_n$
\[ \text{Des}(\pi) := \{ i : \pi(i) > \pi(i + 1) \}, \]
and let
\[ \text{ldes}(\pi) := \max\{ i : i \in \text{Des}(\pi) \} \]
be the last descent of $\pi$, with $\text{ldes}(\pi) := 0$ if $\text{Des}(\pi) = \emptyset$.

Combining Proposition 1.1 with results from [8, 18] one deduces

Corollary 1.2. For every positive integer $n$
\[ \sum_{\pi \in S_n(321)} q^{\text{bl}(\pi)} = \sum_{\pi \in S_n(321)} q^{n-\text{ldes}(\pi)}. \]

A multivariate refinement of Corollary 1.2 is presented in this paper; see Theorem 2.6 below. A new example of a Schur-positive set of permutations follows, addressing a long standing open problem of Gessel and Reutenauer [11] and a more recent one by Sagan and Woo [14].

2. Multivariate equi-distribution

2.1. A bijection.

Definition 2.1. For $1 \leq k \leq n$ denote
\[ \text{Bl}_{n,k} := \{ \pi \in S_n(321) : \text{bl}(\pi) = k \} \]
and
\[ L_{n,k} = \{ \pi \in S_n(321) : \text{ldes}(\pi^{-1}) = k \}. \]

A left-to-right-maxima-preserving bijection from $\text{Bl}_{n,k}$ to $L_{n,n-k}$ is presented in this Subsection.

Definition 2.2. Define maps $f_n : S_n(321) \rightarrow S_n(321)$, recursively, for all $n \geq 1$. For $n = 1$ the definition is obvious, since $S_1(321)$ consists of a unique permutation. For $\pi \in S_n(321)$, $n \geq 2$, the recursive definition of $f_n(\pi)$ depends on $k := \text{bl}(\pi)$ and on the locations of the letters $n-1$ and $n$ in $\pi$.

Distinguish the following three cases:

- **Case A:** $\pi^{-1}(n) = n$, i.e., $n$ is in the last position.
  Then: delete $n$, apply $f_{n-1}$, and insert $n$ at the last position.

- **Case B:** $\pi^{-1}(n-1) < \pi^{-1}(n) < n$, i.e., $n$ is to the right of $n-1$ but not in the last position.
  Then: delete $n$, apply $f_{n-1}$, insert $n$ at the same position as in $\pi$, and multiply on the left by the transposition $(n-k-1, n-k)$.

- **Case C:** $\pi^{-1}(n) < \pi^{-1}(n-1)$, i.e., $n-1$ is to the right of $n$ (and must be the last letter, since $\pi$ is $321$-avoiding).
  Then: let $\pi' := (n-1,n)\pi$, define $f_n(\pi')$ according to case A above, and multiply it on the left by the cycle $(n-k, n-k + 1, \ldots, n)$. 

Remark 2.3. This recursive definition yields a sequence of permutations $(\pi_0, \pi_{n-1}, \ldots, \pi_1)$, starting with $\pi_0 = \pi$. For each $2 \leq i \leq n$, $\pi_{i-1} \in S_{i-1}$ is obtained from $\pi_i \in S_i$ by deleting $i$ from $\pi_i$ (in cases A and B) or by deleting $i$ from $(i-1, i)\pi_i$ (in case C). To recover $f_i(\pi_i)$ from $f_{i-1}(\pi_{i-1})$, the letter $i$ is inserted exactly where it was deleted (for example — in the last position, in cases A and C), and then the permutation is multiplied, on the left, by a suitable cycle.

Example 2.4. Let $\pi = 31254786 \in S_8$, so that $bl(\pi) = 3$ and $ltrMax(\pi) = \{1, 4, 6, 7\}$. The recursive process is illustrated by the following diagram, where the arrow $\pi_i \to \pi_{i-1}$ is decorated by the case and by the corresponding cycle.

\[
\begin{array}{c}
\pi = \pi_8 = 31254786 \xrightarrow{B} \pi_7 = 3125476 \xrightarrow{C} \pi_6 = 312546 \\
\xrightarrow{A} \pi_5 = 31254 \xrightarrow{C} \pi_4 = 3124 \xrightarrow{A} \pi_3 = 312 \\
\xrightarrow{C} \pi_2 = 21 \xrightarrow{(12)} \pi_1 = 1.
\end{array}
\]

$f_1(\pi_1) = 1 \xrightarrow{(12)} f_2(\pi_2) = 21 \xrightarrow{(23)} f_3(\pi_3) = 312 \xrightarrow{4} f_4(\pi_4) = 3124 \\
\xrightarrow{(345)} f_5(\pi_5) = 41253 \xrightarrow{4} f_6(\pi_6) = 412536 \\
\xrightarrow{(4567)} f_7(\pi_7) = 5126374 \xrightarrow{(45)} f_8(\pi) = f_8(\pi_8) = 41263785.$

Note that here one has $ldes(f_8(\pi)^{-1}) = 5 = 8 - bl(\pi)$ and $ltrMax(f_8(\pi)) = \{1, 4, 6, 7\} = ltrMax(\pi)$.

Our main claim is

Theorem 2.5. For each $1 \leq k \leq n$, the map $f_n$ defined above is a left-to-right-maxima-preserving bijection from $Bl_{n,k}$ onto $L_{n,n-k}$.

The complete proof of Theorem 2.5 is given in the full paper version [1].

2.2. Main theorem. Let $ltrMax(\pi) := \{ i : \pi(i) = \max\{\pi(1), \ldots, \pi(i)\} \}$ be the set of left-to-right maxima in a permutation $\pi$. For every $J \subseteq [n]$ let $x^J := \prod_{i \in J} x_i$. Theorem 2.5 implies

**Theorem 2.6.** For every positive integer $n$

\[
\sum_{\pi \in S_n(321)} x^{ltrMax(\pi)} q^{bl(\pi)} = \sum_{\pi \in S_n(321)} x^{ltrMax(\pi^{-1})} q^{n-ldes(\pi)}.
\]

**Remark 2.7.** Theorem 2.6 is reminiscent of the classical Foata-Schützenberger Theorem

\[
\sum_{\pi \in S_n} x^{Des(\pi)} q^{\inv(\pi)} = \sum_{\pi \in S_n} x^{Des(\pi^{-1})} q^{\maj(\pi)}.
\]
Corollary 2.8. For every positive integer \( n \),
\[
\sum_{\pi \in S_n(321)} \chi^{\text{Des}(\pi)} t^{\pi^{-1}(n)} q^{\text{bl}(\pi)} = \sum_{\pi \in S_n(321)} \chi^{\text{Des}(\pi)} t^{\pi^{-1}(n)} q^{n-\text{ides}(\pi^{-1})},
\]

3. An application to Schur-positivity

Given any subset \( A \) of the symmetric group \( S_n \), define the quasi-symmetric function
\[
Q(A) := \sum_{\pi \in A} F_{n, \text{Des}(\pi)},
\]
where \( \text{Des}(\pi) := \{ i : \pi(i) > \pi(i+1) \} \) is the descent set of \( \pi \) and \( F_{n,D} \) (for \( D \subseteq [n-1] \)) are Gessel’s fundamental quasi-symmetric functions; see [1] for more details. The following long-standing problem was first posed in [11].

Problem 3.1. For which subsets \( A \subseteq S_n \) is \( Q(A) \) symmetric?

A symmetric function is called Schur-positive if all the coefficients in its expansion in the basis of Schur functions are nonnegative. Determining whether a given symmetric function is Schur-positive is a major problem in contemporary algebraic combinatorics [17].

Call a subset \( A \subseteq S_n \) Schur-positive if \( Q(A) \) is symmetric and Schur-positive. Classical examples of Schur-positive sets of permutations include inverse descent classes [10], Knuth classes [10], conjugacy classes [11, Theorem 5.5], and permutations with a fixed inversion number [2, Prop. 9.5].

New constructions of Schur-positive sets of permutations were described in [9] and [14]. Inspired by these examples, Sagan and Woo raised the problem of finding Schur-positive pattern-avoiding sets [14].

The goal of this paper is to present a new example of a Schur-positive set of permutations which involves pattern-avoidance: the set of 321-avoiding permutations having a prescribed number of blocks. We shall state that more explicitly.

For an integer partition \( \lambda \) of \( n \), let \( \chi^\lambda \) and \( s_\lambda \) be the irreducible \( S_n \)-character and the Schur function indexed by \( \lambda \), respectively. Recall the Frobenius characteristic map \( \text{ch} \), from class functions on \( S_n \) to symmetric functions, defined by \( \text{ch}(\chi^\lambda) = s_\lambda \) and extended by linearity. Corollary 2.8 implies

Theorem 3.2. For any \( 1 \leq k \leq n \), the set \( \text{Bl}_{n,k} = \{ \pi \in S_n(321) \mid \text{bl}(\pi) = k \} \) is Schur-positive. In fact, for \( 1 \leq k \leq n-1 \)
\[
Q(\text{Bl}_{n,k}) = \text{ch}(\chi^{(n-1,n-k)} \downarrow_{S_n} S_{n-k-1})
\]
while for \( k = n \)
\[
Q(\text{Bl}_{n,n}) = \text{ch}(\chi^{(n)}) = s(n).
\]
4. Final remarks

Time permitting, we shall also discuss applications and implications of the above results to Hilbert series of certain polynomial rings and to the search for Schur-positive statistics on pattern-avoiding sets.

References


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