BLOCK NUMBERS, 321-AVOIDANCE AND SCHUR-POSITIVITY

RON M. ADIN, ELI BAGNO AND YUVAL ROICHMAN

1. The block number of a permutation

1.1. Definitions. Direct sums and the block decomposition of permutations appear naturally in the study of pattern-avoiding classes [3, 4].

Let $\pi \in \mathcal{S}_m$ and $\sigma \in \mathcal{S}_n$. The *direct sum* of π and σ is the permutation $\pi \oplus \sigma \in \mathcal{S}_{m+n}$ defined by

$$\pi \oplus \sigma := \begin{cases} \pi(i), & \text{if } i \le m; \\ \sigma(i-m)+m, & \text{otherwise.} \end{cases}$$

For example, if $\pi = 312$ and $\sigma = 2413$ then $\pi \oplus \sigma = 3125746$; see Figure 1.

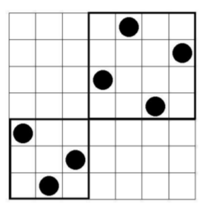

Figure 1. The permutation $312 \oplus 2413 = 3125746$

A nonempty permutation which is not the direct sum of two nonempty permutations is called $\oplus$-*irreducible*. Each permutation π can be written uniquely as a direct sum of $\oplus$-irreducible ones, called the *blocks* of π; their number, denoted by $\mathrm{bl}(\pi)$, is the *block number* of π. Equivalently,

$$\mathrm{bl}(\pi) = |\{1 \le i \le n \,:\, (\forall j \le i)\, \pi(j) \le i\}|.$$

1.2. Counting 321-avoiding permutations by block number. Recall the n-th Catalan number, $C_n := \frac{1}{n+1}\binom{2n}{n}$, and its generating function $c(x) := \sum_{n=0}^{\infty} C_n x^n$. For each $0 \le k \le n$, the *n-th k-fold Catalan number* $C_{n,k}$ is the coefficient of x^n in $(xc(x))^k$. These numbers are also called *ballot numbers*, and form the *Catalan triangle* [15, A009766].

A permutation $\pi \in \mathcal{S}_n$ is 321-avoiding if the sequence $(\pi(1), \ldots, \pi(n))$ contains no decreasing subsequence of length 3. Denote by $\mathcal{S}_n(321)$ the set of 321-avoiding permutations in $\mathcal{S}_n$.

Proposition 1.1. [1] *For any fixed positive integer k, the ordinary generating function for the number of 321-avoiding permutations in $\mathcal{S}_n$ with exactly k blocks is $(xc(x))^k$.*

Recall the *descent set* of a permutation $\pi \in \mathcal{S}_n$

$$\mathrm{Des}(\pi) \coloneqq \{i : \ \pi(i) > \pi(i+1)\},$$

and let

$$\mathrm{ldes}(\pi) \coloneqq \max\{i \ : \ i \in \mathrm{Des}(\pi)\}$$

be the *last descent* of π, with $\mathrm{ldes}(\pi) \coloneqq 0$ if $\mathrm{Des}(\pi) = \varnothing$.

Combining Proposition 1.1 with results from [8, 18] one deduces

Corollary 1.2. *For every positive integer n*

$$\sum_{\pi \in \mathcal{S}_n(321)} q^{\mathrm{bl}(\pi)} = \sum_{\pi \in \mathcal{S}_n(321)} q^{n-\mathrm{ldes}(\pi)}.$$

A multivariate refinement of Corollary 1.2 is presented in this paper; see Theorem 2.6 below. A new example of a Schur-positive set of permutations follows, addressing a long standing open problem of Gessel and Reutenauer [11] and a more recent one by Sagan and Woo [14].

2. Multivariate equi-distribution

2.1. A bijection.

Definition 2.1. For $1 \le k \le n$ denote

$$Bl_{n,k} \coloneqq \{\pi \in \mathcal{S}_n(321) : \ \mathrm{bl}(\pi) = k\}$$

and

$$L_{n,k} = \{\pi \in \mathcal{S}_n(321) : \ \mathrm{ldes}(\pi^{-1}) = k\}.$$

A left-to-right-maxima-preserving bijection from $Bl_{n,k}$ to $L_{n,n-k}$ is presented in this Subsection.

Definition 2.2. Define maps $f_n : \mathcal{S}_n(321) \to \mathcal{S}_n(321)$, recursively, for all $n \ge 1$. For $n = 1$ the definition is obvious, since $\mathcal{S}_1(321)$ consists of a unique permutation. For $\pi \in \mathcal{S}_n(321)$, $n \ge 2$, the recursive definition of $f_n(\pi)$ depends on $k \coloneqq \mathrm{bl}(\pi)$ and on the locations of the letters $n-1$ and n in π. Distinguish the following three cases:

- **Case A:** $\pi^{-1}(n) = n$, i.e., n is in the last position.
  Then: delete n, apply f_{n-1}, and insert n at the last position.
- **Case B:** $\pi^{-1}(n-1) < \pi^{-1}(n) < n$, i.e., n is to the right of $n-1$ but not in the last position.
  Then: delete n, apply f_{n-1}, insert n at the same position as in π, and multiply on the left by the transposition $(n-k-1, n-k)$.
- **Case C:** $\pi^{-1}(n) < \pi^{-1}(n-1)$, i.e., $n-1$ is to the right of n (and must be the last letter, since π is 321-avoiding).
  Then: let $\pi' \coloneqq (n-1, n)\pi$, define $f_n(\pi')$ according to case A above, and multiply it on the left by the cycle $(n-k, n-k+1, ..., n)$.

Remark 2.3. *This recursive definition yields a sequence of permutations* $(\pi_n, \pi_{n-1}, \ldots, \pi_1)$*, starting with* $\pi_n = \pi$*. For each* $2 \le i \le n$*,* $\pi_{i-1} \in \mathcal{S}_{i-1}$ *is obtained from* $\pi_i \in \mathcal{S}_i$ *by deleting* i *from* π_i *(in cases A and B) or by deleting* i *from* $(i-1,i)\pi_i$ *(in case C). To recover* $f_i(\pi_i)$ *from* $f_{i-1}(\pi_{i-1})$*, the letter* i *is inserted exactly where it was deleted (for example — in the last position, in cases A and C), and then the permutation is multiplied, on the left, by a suitable cycle.*

Example 2.4. Let $\pi = 31254786 \in \mathcal{S}_8$, so that $\mathrm{bl}(\pi) = 3$ and $\mathrm{ltrMax}(\pi) = \{1,4,6,7\}$. The recursive process is illustrated by the following diagram, where the arrow $\pi_i \to \pi_{i-1}$ is decorated by the case and by the corresponding cycle.

$$\begin{aligned}
\pi = \pi_8 = 31254786 \quad &\xrightarrow[(45)]{B} \quad \pi_7 = 3125476 \xrightarrow[(4567)]{C} \pi_6 = 312546 \\
&\xrightarrow{A} \quad \pi_5 = 31254 \xrightarrow[(345)]{C} \pi_4 = 3124 \xrightarrow{A} \pi_3 = 312 \\
&\xrightarrow[(23)]{C} \quad \pi_2 = 21 \xrightarrow[(12)]{C} \pi_1 = 1.
\end{aligned}$$

$$\begin{aligned}
f_1(\pi_1) = 1 \quad &\xrightarrow{(12)} \quad f_2(\pi_2) = 21 \xrightarrow{(23)} f_3(\pi_3) = 312 \longrightarrow f_4(\pi_4) = 3124 \\
&\xrightarrow{(345)} \quad f_5(\pi_5) = 41253 \longrightarrow f_6(\pi_6) = 412536 \\
&\xrightarrow{(4567)} \quad f_7(\pi_7) = 5126374 \xrightarrow{(45)} f_8(\pi) = f_8(\pi_8) = 41263785.
\end{aligned}$$

Note that here one has $\mathrm{ldes}(f_8(\pi)^{-1}) = 5 = 8 - \mathrm{bl}(\pi)$ and $\mathrm{ltrMax}(f_8(\pi)) = \{1,4,6,7\} = \mathrm{ltrMax}(\pi)$.

Our main claim is

Theorem 2.5. *For each* $1 \le k \le n$*, the map* f_n *defined above is a left-to-right-maxima-preserving bijection from* $Bl_{n,k}$ *onto* $L_{n,n-k}$*.*

The complete proof of Theorem 2.5 is given in the full paper version [1].

2.2. **Main theorem.** Let

$$\mathrm{ltrMax}(\pi) \coloneqq \{i : \pi(i) = \max\{\pi(1), \ldots, \pi(i)\}\}$$

be the set of *left-to-right maxima* in a permutation π. For every $J \subseteq [n]$ let $\mathbf{x}^J \coloneqq \prod_{i \in J} x_i$. Theorem 2.5 implies

Theorem 2.6. *For every positive integer* n

$$\sum_{\pi \in \mathcal{S}_n(321)} \mathbf{x}^{\mathrm{ltrMax}(\pi^{-1})} q^{\mathrm{bl}(\pi)} = \sum_{\pi \in \mathcal{S}_n(321)} \mathbf{x}^{\mathrm{ltrMax}(\pi^{-1})} q^{n-\mathrm{ldes}(\pi)}.$$

Remark 2.7. *Theorem 2.6 is reminiscent of the classical Foata-Schützenberger Theorem*

$$\sum_{\pi \in \mathcal{S}_n} \mathbf{x}^{\mathrm{Des}(\pi^{-1})} q^{\mathrm{inv}(\pi)} = \sum_{\pi \in \mathcal{S}_n} \mathbf{x}^{\mathrm{Des}(\pi^{-1})} q^{\mathrm{maj}(\pi)}.$$

Corollary 2.8. *For every positive integer n,*

$$\sum_{\pi\in\mathcal{S}_n(321)} \mathbf{x}^{\mathrm{Des}(\pi)} t^{\pi^{-1}(n)} q^{\mathrm{bl}(\pi)} = \sum_{\pi\in\mathcal{S}_n(321)} \mathbf{x}^{\mathrm{Des}(\pi)} t^{\pi^{-1}(n)} q^{n-\mathrm{ldes}(\pi^{-1})}.$$

3. An application to Schur-positivity

Given any subset A of the symmetric group $\mathcal{S}_n$, define the quasi-symmetric function

$$\mathcal{Q}(A) := \sum_{\pi\in A} \mathcal{F}_{n,\mathrm{Des}(\pi)},$$

where $\mathrm{Des}(\pi) := \{i : \ \pi(i) > \pi(i+1)\}$ is the *descent set* of π and $\mathcal{F}_{n,D}$ (for $D \subseteq [n-1]$) are Gessel's *fundamental quasi-symmetric functions;* see [1] for more details. The following long-standing problem was first posed in [11].

Problem 3.1. For which subsets $A \subseteq \mathcal{S}_n$ is $\mathcal{Q}(A)$ symmetric?

A symmetric function is called *Schur-positive* if all the coefficients in its expansion in the basis of Schur functions are nonnegative. Determining whether a given symmetric function is Schur-positive is a major problem in contemporary algebraic combinatorics [17].

Call a subset $A \subseteq \mathcal{S}_n$ *Schur-positive* if $\mathcal{Q}(A)$ is symmetric and Schur-positive. Classical examples of Schur-positive sets of permutations include inverse descent classes [10], Knuth classes [10], conjugacy classes [11, Theorem 5.5], and permutations with a fixed inversion number [2, Prop. 9.5].

New constructions of Schur-positive sets of permutations were described in [9] and [14]. Inspired by these examples, Sagan and Woo raised the problem of finding Schur-positive pattern-avoiding sets [14].

The goal of this paper is to present a new example of a Schur-positive set of permutations which involves pattern-avoidance: the set of 321-avoiding permutations having a prescribed number of blocks. We shall state that more explicitly.

For an integer partition λ of n, let χ^λ and s_λ be the irreducible $\mathcal{S}_n$-character and the Schur function indexed by λ, respectively. Recall the *Frobenius characteristic map* ch, from class functions on $\mathcal{S}_n$ to symmetric functions, defined by $\mathrm{ch}(\chi^\lambda) = s_\lambda$ and extended by linearity. Corollary 2.8 implies

Theorem 3.2. *For any $1 \le k \le n$, the set $Bl_{n,k} = \{\pi \in \mathcal{S}_n(321) \mid bl(\pi) = k\}$ is Schur-positive. In fact, for $1 \le k \le n-1$*

$$\mathcal{Q}(Bl_{n,k}) = \mathrm{ch}(\chi^{(n-1,n-k)} \downarrow^{\mathcal{S}_{2n-k-1}}_{\mathcal{S}_n})$$

while for $k = n$

$$\mathcal{Q}(Bl_{n,n}) = \mathrm{ch}(\chi^{(n)}) = s_{(n)}.$$

4. Final remarks

Time permitting, we shall also discuss applications and implications of the above results to Hilbert series of certain polynomial rings and to the search for Schur-positive statistics on pattern-avoiding sets.

References

[1] R. M. Adin, E. Bagno and Y. Roichman, *Block decomposition of permutations and Schur-positivity*, `arXiv:1611.06979`.

[2] R. M. Adin and Y. Roichman, *Matrices, characters and descents*, Linear Algebra Appl. **469** (2015), 381–418.

[3] M. Albert and M. Atkinson, *Simple permutations and pattern restricted permutations*, Discrete Math. **300** (2005), 1–15.

[4] M. H. Albert, M. Atkinson and V. Vatter, *Subclasses of the separable permutations*, Bull. Lond. Math. Soc. **43** (2011), 859–870.

[5] J.-C. Aval, F. Bergeron and N. Bergeron, *Ideals of quasi-symmetric functions and super-covariant polynomials for S_n*, Adv. Math. **181** (2004), 353–367.

[6] M. Bóna, Combinatorics of Permutations, With a foreword by Richard Stanley, Discrete Math. and its Applications, Chapman & Hall/CRC, Boca Raton, FL, 2004.

[7] E. Catalan, *Sur les nombres de Segner*, Rend. Circ. Mat. Palermo **1** (1887), 190–201.

[8] S. Connolly, Z. Gabor and A. P. Godbole, *The location of the first ascent in a* 123-*avoiding permutation*, Integers **15** (2015), Paper A13, 8 pages.

[9] S. Elizalde and Y. Roichman, *Schur-positive sets of permutations via products and grid classes*, J. Algeb. Combin., to appear; `arXiv:1509.00045`.

[10] I. M. Gessel, *Multipartite P-partitions and inner products of Schur functions*, Contemp. Math. **34** (1984), 289–302.

[11] I. M. Gessel and C. Reutenauer, *Counting permutations with given cycle structure and descent set*, J. Combin. Theory Ser. A **64** (1993), 189–215.

[12] M. Joseph and T. Roby, personal communication.

[13] B. E. Sagan, The symmetric group: Representations, combinatorial algorithms, and symmetric functions, 2nd edition, Graduate Texts in Math., no. 203, Springer-Verlag, New York, 2001.

[14] B. E. Sagan, *Pattern avoidance and quasisymmetric functions*, The 13th International Permutation Patterns Conference, London, UK, 2015. `https://sites.google.com/site/pp2015london/slides`

[15] N. J. A. Sloane, The On-line Encyclopedia of Integer Sequences.

[16] R. P. Stanley, Enumerative Combinatorics, Vol. 2, Cambridge Studies in Adv. Math., no. 62, Cambridge Univ. Press, Cambridge, 1999.

[17] R. P. Stanley, *Positivity problems and conjectures in algebraic combinatorics*, in: Mathematics: Frontiers and Perspectives (V. Arnold, M. Atiyah, P. Lax, and B. Mazur, eds.), American Math. Society, Providence, RI, 2000, pp. 295–319.

[18] S. Tedford, *Combinatorial interpretations of convolutions of the Catalan numbers*, Integers **11** (2011), Paper A3, 10 pages.

Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel, Email: `radin@math.biu.ac.il`

Department of Applied Mathematics, Jerusalem college of Technology, Jerusalem, Israel, Email: `bagnoe@jct.ac.il`

Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel, Email: `yuvalr@math.biu.ac.il`