Staircases, dominoes and leaves: Bounds on gr(Av(1324))

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This talk is based on joint work with
Robert Brignall, Andrew Elvey Price and Jay Pantone.

We present improved upper and lower bounds on the growth rate of Av(1324). Our results rely on the following structural characterization of 1324-avoiders.

Proposition 1. *The class of 1324-avoiders is contained in the infinite generalized grid class consisting of a decreasing staircase whose upper right cells contain* Av(213) *and whose lower left cells contain* Av(132)*.*

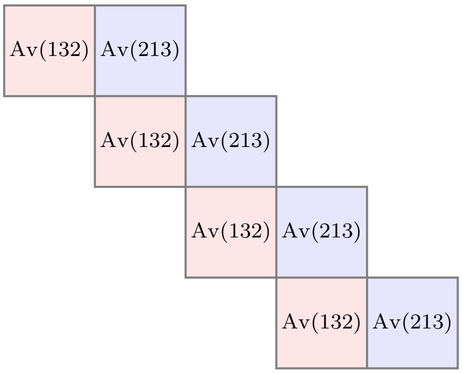

Figure 1: Part of the infinite staircase containing Av(1324)

A *domino* is a 1324-avoiding *gridded permutation* in the two-celled generalized grid class in which Av(213) is juxtaposed above Av(132).

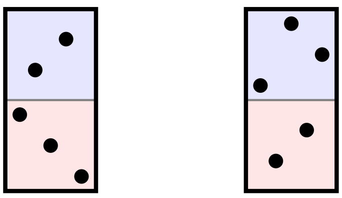

Figure 2: Two small dominoes

Dominoes may be enumerated exactly by exploiting a bijection with certain arch systems, and solving the resulting functional equation using iterated discriminants.

Proposition 2. *The number of dominoes on n points is* $\dfrac{2(3n+3)!}{(n+2)!(2n+3)!}$*. The growth rate of this sequence is* $27/4$*.*

This sequence is A000139 in OEIS, which also counts both West-two-stack-sortable permutations and rooted nonseparable planar maps. No bijection between dominoes and either of these is known.

We establish the following upper bound on gr(Av(1324)) by exhibiting an injection which maps a gridded 1324-avoider to a pair consisting of a domino and a binary sequence that records the interleaving of points in adjacent dominoes in the gridding.

Proposition 3. *The growth rate of* Av(1324) *is at most* $27/2 = 13.5$*.*

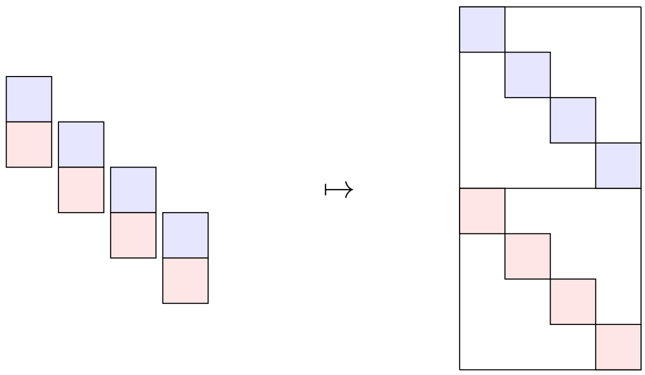

Figure 3: Mapping a gridded 1324-avoider to a domino

To establish a lower bound, the decomposition illustrated in Figure 4 is used. Dominoes (blue and red cells) are joined by yellow and green *connecting cells*, which contain a specified number of skew-indecomposable components. By positioning the points in the dominoes *between* these components, 1324 is avoided. This yields the following lower bound.

Proposition 4. *The growth rate of* Av(1324) *is at least* $81/8 = 10.125$*.*

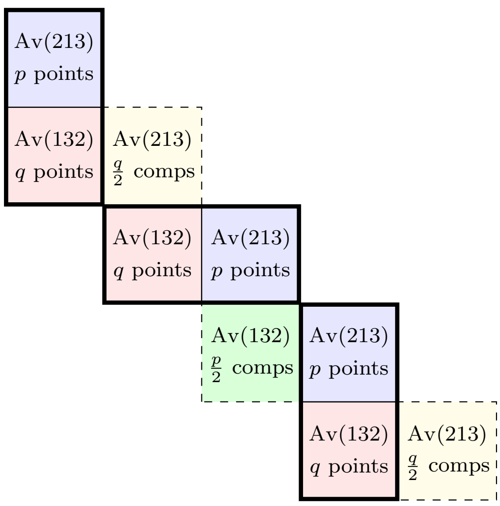

Figure 4: Decomposing a staircase into dominoes and connecting cells

Further analysis of the structure of dominoes enables us to improve this bound. A *leaf* in a domino is a point which is either a right-to-left maximum in the upper cell, or a left-to-right minimum in the lower cell. The approach used in Proposition 2 can be extended to enumerate the leaves.

Proposition 5. *The expected number of leaves in a domino on n points is asymptotically $5n/9$.*

To avoid 1324, it is not necessary for domino leaves to be positioned between the components of connecting cells (see Figure 5). Requiring non-leaves to occur between the components while permitting leaves to be arbitrarily interleaved with the points in the connecting cells enables us to establish an increased lower bound.

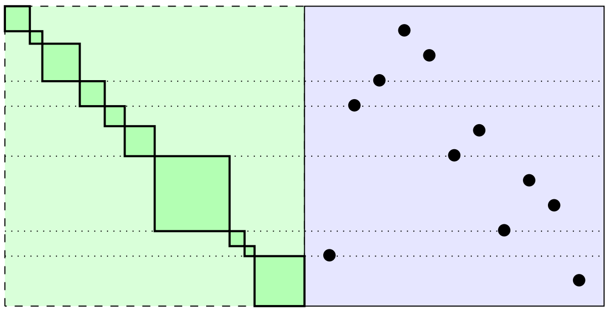

Figure 5: Non-leaves positioned between skew-indecomposable components

The non-leaves in a domino cell divide the cell into strips. Further analysis of arch systems and another application of the technique used in Proposition 2 determines how many of these are empty.

Proposition 6. *The expected number of strips that contain no leaves occurring in a domino on n points is asymptotically $5n/27$.*

Combining Propositions 5 and 6 with a technical lemma concerning log-concave sequences enables us to determine the "worst case" distribution of leaves across the strips in a domino. Modelling this with a system of 27 functional equations then yields the following lower bound.

Proposition 7. *The growth rate of* $\mathrm{Av}(1324)$ *is at least* $10.2710129\ldots$*, the root of a polynomial of degree 104.*