Asymptotics of Pattern Avoidance in the Klazar Set Partition and Permutation-Tuple Settings

Permutation Patterns 2017 Abstract

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We first introduce set partitions and the notion of Klazar-type set partition pattern avoidance.

Definition. A *set partition* is a partition of the set $[n]$ for some n into any number of nonempty sets, where the order within the partition is irrelevant. We call these sets *blocks* of the partition. The number of set partitions of $[n]$ is the *Bell number* B_n.

Often, when writing specific set partitions, we will write the partition $[n] = S_1 \cup \cdots \cup S_k$ as $S_1/\cdots/S_k$, where the S_i are in increasing order of smallest element, and are written as strings of numbers from least to greatest; for example, $[5] = \{2, 4\} \cup \{1, 3, 5\}$ would be written 135/24. To carry over our notions of avoidance, we define pattern containment on set partitions.

Definition. Let π and π' be set partitions of $[n]$ and $[m]$, respectively. We say that π contains (respectively avoids) π' if there exists (respectively does not exist) a strictly increasing function $f : [m] \to [n]$ such that for any $i, j \in [m]$, i and j are in the same block of π' if and only if $f(i)$ and $f(j)$ are in the same block of π.

(Note that this is distinct from RGF-type set partition avoidance as studied in [5], where, for example, 145/23 would avoid 12/34, as in RGF-type containment the order of blocks must be preserved.)

We will be concerned with the asymptotics of pattern classes of set partitions. In analogy to the permutation case, where all pattern classes grow as $n!$ or are bounded above by an exponential, we find that we can similarly classify the growth rate of pattern classes of set partitions to within an exponential factor. Specifically, we have the following.

Theorem 1. Let $\mathcal{P}$ be a nonempty pattern class of set partitions, and as usual let $\mathcal{P}_n$ be the partitions of $[n]$ in $\mathcal{P}$. Then one of the following is true.

1. $\mathcal{P}$ contains every set partition; that is, $|\mathcal{P}_n| = B_n$.

2. There exists a positive integer d and real constants $c_2 > c_1 > 0$ such that for every n,

$$c_1^n n^{n\left(1-\frac{1}{d}\right)} \leq \mathcal{P}_n \leq c_2^n n^{n\left(1-\frac{1}{d}\right)}$$

To explain when the pattern class falls into a particular asymptotic range, we relate partitions to tuples of permutations and define the permutability statistic.

Definition. Let $\sigma_1, \ldots, \sigma_d \in S_n$ be permutations. We then define $[\sigma_1, \ldots, \sigma_d]$ to be the set partition of $[(d+1)n]$ containing n blocks $B_1, \ldots, B_n$, where $B_i = \{i, n + \sigma_1(i), 2n + \sigma_2(i), \ldots, dn + \sigma_d(i)\}$.

Definition. Let π be a set partition of $[n]$. The *permutability* of π, denoted $\mathrm{pm}(\pi)$, is the minimal d such that there exists $m \in \mathbb{Z}^+$ and $\sigma_1, \ldots, \sigma_d \in S_m$ such that $[\sigma_1, \ldots, \sigma_d]$ contains π.

We now have the terminology to describe which pattern classes correspond to which d in Theorem 1: d is the smallest permutability of a set partition not in $\mathcal{P}$. (We ignore the case where this is 0–this corresponds also to $d = 1$.)

We now outline the methods of proof of the lower and upper bounds. For the lower bound, note that by assumption, all partitions of permutability $d-1$ are in $\mathcal{P}$. This will include all partitions of the form $[\sigma_1, \ldots, \sigma_{d-1}]$. On $[dn]$, there are $(n!)^{d-1}$ such partitions, which already gives the desired lower bound.

As usual, the upper bound is far more difficult; we will simply describe some of the techniques and lemmas involved. It suffices to show the result for classes given by avoiding one element $[\sigma_1, \ldots, \sigma_d]$, as every pattern class with the correct asymptotic is contained in such a class.

As with permutations, where often it is necessary to generalize to $0-1$-matrices, we may generalize from set partitions to ordered hypergraphs. Just as $0-1$ matrices correspond to bipartite graphs, with permutation matrices corresponding to matchings or 1-regular bipartite graphs, set partitions correspond to 1-regular ordered hypergraphs. In particular, the edges of a 1-regular ordered hypergraph give a partition of the vertex set $[n]$.

This implies that we should look for some Füredi-Hajnal type bound on hypergraphs that avoid a hypergraph of the desired form. We first make several definitions.

Definition. The ordered hypergraph G *contains* (respectively *avoids*) the ordered hypergraph H if there exists (respectively does not exist) an order-preserving injection $i_V : V(H) \to V(G)$ and and an injection $i_E : E(H) \to E(G)$ that are compatible, in the sense that if $v \in e \in E(H)$, then $i_V(v) \in i_E(e)$. A d-permutation hypergraph is a hypergraph corresponding to some set partition $[\sigma_1, \ldots, \sigma_{d-1}]$.

Note that the definition ensures that a d-permutation hypergraph is d-uniform. We may now state our Füredi-Hajnal type lemma.

Lemma 1. *For an ordered hypergraph G, define $i(G) := \sum_{E \in E(G)} |E|$. Let H be a d-permutation hypergraph. Then there exists a constant c such that for all $n \in \mathbb{Z}^+$ and ordered hypergraphs G on $[n]$ avoiding H,*

$$i(G) \leq cn^{d-1}.$$

The proof of the lemma is a technical induction, first proving a stronger statement for t-uniform hypergraphs and then using that to obtain the result for general graphs. Note that Lemma 1 is a generalization of two results of Klazar and Marcus in [4], one of which is simply the $d = 2$ case of Lemma 1, and the other deals with d-dimensional $0 - 1$ matrices, which give a subset of the cases when G is d-uniform.

Remark. This result is quite interesting in its own right, and prompts the question of determining asymptotics of $\max_{\substack{G \text{ avoids } H \\ V(G)=[n]}} i(G)$ for various H (or to generalize to pattern classes of ordered hypergraphs). From Lemma 1 it is not difficult to see that if H is 1-regular, corresponding to some set partition π, then this is within a constant on either side of $n^{\mathrm{pm}(\pi)}$. For non-1-regular hypergraphs, the answer may be more complicated; for example, if G is restricted to be a graph, Klazar shows in [3] that it is possible to achieve a function that grows very slightly faster than linearly.

From Lemma 1 it is a routine recursive derivation that the *number* of ordered hypergraphs G on $[n]$ that avoid some d-permutation hypergraph H is bounded above asymptotically by $c'^{n^{d-1}}$ for some constant c'. Unfortunately, this is far greater than the desired $n^{n\left(1-\frac{1}{d-1}\right)}$, as we are now considering general hypergraphs and not just set partitions.

However, a final trick finishes the solution; given a set partition in the form of a 1-regular hypergraph G on $[n]$, we may divide the n vertices into s intervals of size $\frac{n}{s}$. For each edge E of the original graph, we can create an edge E' of the new graph containing exactly those vertices corresponding to intervals in which E contained at least one vertex. (We then delete duplicate edges.) If the resulting graph is G', we know that we only have $c'^{s^{d-1}}$ choices for G'. Then, we may bound how many set partitions correspond to a particular choice of G'. Optimizing for s leads after some algebra to the desired bound.

We may now apply this result to permutation-tuple pattern avoidance, which we now define.

Definition. Fix $d \in \mathbb{Z}^+$. Let $\sigma_1, \ldots, \sigma_d \in S_n$ and $\sigma'_1, \ldots, \sigma'_d \in S_m$ be permutations. Then the d-tuple $(\sigma_1, \ldots, \sigma_d)$ contains (respectively avoids) the d-tuple $(\sigma_1, \ldots, \sigma_d)$ if there exist (respectively do not exist) indices $i_1, \ldots, i_m \in [n]$ with $i_1 < \cdots < i_m$ satisfying the property that for any j, $\sigma_j(i_1) \cdots \sigma_j(i_m)$ has the same relative ordering as $\sigma'_j(1) \cdots \sigma'_j(m)$.

In other words, a d-tuple T_1 contains another d-tuple T_2 if and only if each permutation in T_1 contains the corresponding permutation in T_2 at the same location.

It is not difficult to see that d-permutation-tuple avoidance is equivalent to set partition avoidance where the set partitions are restricted to the form $[\sigma_1, \ldots, \sigma_d]$. That is, $[\sigma_1, \ldots, \sigma_d]$ contains $[\sigma'_1, \ldots, \sigma'_d]$ if and only if $(\sigma_1, \ldots, \sigma_d)$ contains $(\sigma'_1, \ldots, \sigma'_d)$.

This immediately gives an upper bound on the number of elements of S_n^d that avoid a particular d-tuple $(\sigma_1, \ldots, \sigma_d)$: it is at most the number of set partitions of $[(d+1)n]$ that avoid $[\sigma_1, \ldots, \sigma_d]$. By our earlier result this is at most $c^n n^{\left(\frac{d^2-1}{d}\right)n}$ for some constant c.

Intriguingly, the lower bound in this case also holds (this follows quickly from results in [1])–that is, the number of elements of S_n^d that avoid $(\sigma_1, \ldots, \sigma_d)$ is at least $c^n n^{\left(\frac{d^2-1}{d}\right)n}$ for some constant $c > 0$. In sum, we have the following result.

Theorem 2. Fix $d, m \in \mathbb{Z}^+$, and $\sigma_1, \ldots, \sigma_d \in S_m$. For a particular n, let $S_n^d(\sigma_1, \ldots, \sigma_d)$ be the set of d-tuples of permutations in S_n that avoid $(\sigma_1, \ldots, \sigma_d)$. Then there exist constants $c_2 > c_1 > 0$ so that for all n,

$$c_1^n n^{\left(\frac{d^2-1}{d}\right)n} \leq \left|S_n^d(\sigma_1, \ldots, \sigma_d)\right| < c_2^n n^{\left(\frac{d^2-1}{d}\right)n}.$$

When $d = 1$, Theorem 2 simply returns Stanley-Wilf. For higher d this is highly nontrivial even in simple cases–for example, there does not seem to be a closed form for the number of pairs of permutations in S_n that avoid $(12, 12)$. Even the asymptotics here are not well bounded: by Theorem 2 we know that it will be bounded within an exponential of $(n!)^{\frac{3}{2}}$ (note that in any of these theorems, by Stirling Approximation, we can replace n^n by $n!$), but the base of that exponential factor is only currently known (to the author's knowledge) to be bounded between 1 and $3\sqrt{3(3\log 3 - 4\log 2)} \approx 3.76$, as proven in [2]. (This special case is the sequence of number of pairs $(\sigma_1, \sigma_2) \in S_n^2$ with $\sigma_1 \leq \sigma_2$ in the weak Bruhat order.)

There are several questions that remain here; most immediately, in the case of d-tuples, we have not shown any classification of growth rates of pattern classes like the one we showed in the set partition case. Of course, the pattern class of every d-tuple (that is, basis size 0) grows as $(n!)^d$ (or, to use the kinds of expressions above, within exponentially of n^{dn}); Theorem 2 shows that any pattern class of d-tuples of permutations with exactly one (nontrivial) basis element grows within exponentially of $n^{\left(\frac{d^2-1}{d}\right)n}$. Other growth orders are possible; we can take the product of a pattern class of d_1-tuples and a pattern class of d_2-tuples to get a pattern class of $d_1 + d_2$-tuples. This immediately implies that we may obtain growth rates within exponential of n^{cn} where $c = d - \sum_{i=1}^{k} \frac{1}{n_i}$, where $k, n_1, \ldots, n_k \in \mathbb{Z}^+$ with $n_1 + \cdots + n_k \leq d$. Are there any other posible growth rates? It is not clear what we might expect from, for example, an arbitrary pattern class of pairs of permutations (so $d = 2$) with basis size 2.

The permutability statistic of partitions does not seem to be distributed according to other known statistics of partitions. Determining this distribution might help specify the constants in Theorem 1, as $\mathcal{P}_n$ is bounded below by the number of set partitions of $[n]$ permutability at most $d - 1$.

References

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