Permutations sortable by deques and by two stacks in parallel

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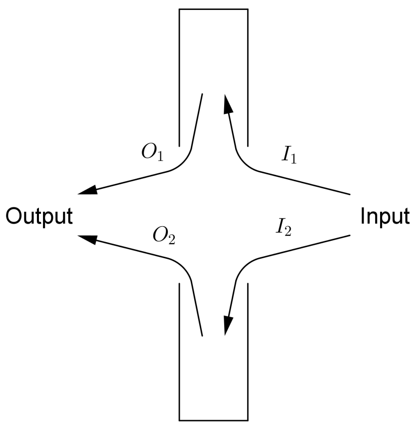
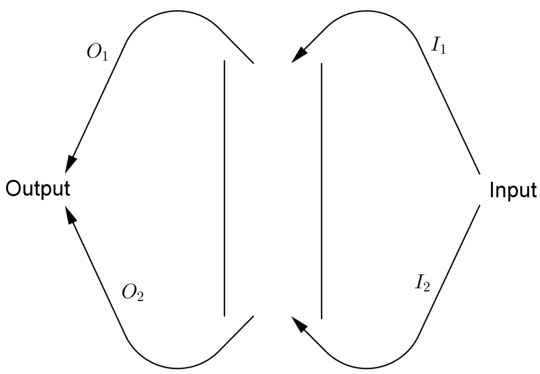
This talk is based on joint work with Tony Guttmann.

In his seminal work *The Art of Computer Programming* [2], Knuth was the first to consider a number of classic data structures from the point of view of the permutations they could produce from the identity permutation, or equinumerously, the permutations which the data structure can sort. Famously, he noticed that the permutations obtainable using a single stack are exactly those which avoid the pattern 312. Knuth posed the same question for a number of other data structures, in particular, he asked how many permutations of each length can be obtained using a double ended queue (deque) or two stacks in parallel (tsip).

In [1], Albert and Bousquet-Melou characterised the counting function $P(t)$ for tsip-sortable permutations in terms of a generating function $Q(u, a)$ for weighted quarter plane loops. This solves the enumeration problem for tsip-sortable permutations in the sense that P is characterised by a system of functional equations. Using a similar method to Albert and Bousquet-Melou, we present a solution to the problem of enumerating deque-sortable permutations. Our solution takes the form of a simple relationship between the counting function $D(t)$ for deque-sortable permutations and the counting function $P(t)$.





The allowed operations for two stacks in parallel (left) and a deque (right)

A deque-sortable or tsip-sortable permutation can be classified by its operation sequence, that is, a sequence of moves I_1, I_2, O_1, O_2 which sorts the permutation. We will call an operation sequence for a double ended queue a deque-word, and we call an operation sequence for two stacks in parallel a tsip-word. Since many operation sequences correspond to the same permutation, we proceed by defining canonical deque-words, which are in bijection with the permutations they sort.

A deque-word is called canonical if it has the following properties:

1. The subwords I_1O_2 and I_2O_1 are forbidden
2. Any subword of the operation sequence which is a tsip-word begins with I_1.
3. When the deque contains at most 1 element only the moves I_1 and O_1 are allowed, not I_2 or O_2.

This mirrors the characterisation of canonical tsip-words given by Albert and Bousquet-Melou, which are defined by satisfying the first two conditions.

By enumerating canonical deque-words, we derive the following relationship between the counting function $D(t)$ for deque-sortable permutations, and the counting function $P(t)$ for tsip-sortable permutations:

$$2D(t) = 2 + t + 2Pt - 2Pt^2 - t\sqrt{1 - 4P + 4P^2 - 8P^2t + 4P^2t^2 - 4Pt}.$$

Despite the fact that both problems are in some sense solved, it is still not proven that the radii of convergence t_p of $P(t)$ and t_d of $D(t)$ are equal. From the expression above, we can immediately deduce that if

$$1 - 4P + 4P^2 - 8P^2t + 4P^2t^2 - 4Pt > 0$$

for all $t \in [0, t_p)$, then $t_d = t_p$. It is conjectured, however, that the expression above is equal to 0 at $t = t_p$, so it is not possible to prove the inequality by simply approximating the left hand side. Since the function P has a simple characterisation in terms of a generating function $Q(u, a)$ for weighted quarter plane loops, the question of whether $t_d = t_p$ can be reduced to questions about Q.

Define a quarter plane loop to be a lattice path in the nonnegative quadrant which starts and ends at $(0, 0)$. A weighted quarter plane loop is then defined by attaching a weight a to NW and ES corners. Then the generating function $Q(u, a)$ for weighted quarter plane loops is given by

$$Q(u, a) = \sum q_{n,m} u^n a^m,$$

where $q_{n,m}$ is the number of quarter plane loops with length $2n$ which contain m weighted corners. Albert and Bousquet-Melou showed that the generating function $P(t)$ is characterised by the equation

$$Q\left(\frac{1}{P} - 1, \frac{tP^2}{(1 - 2P)^2}\right) = 2P - 1.$$

We show that the following three conjectures from [1] imply that $t_d = t_p$.

Conjecture 1. *The series $Q(a, u)$ is $(a + 1)$-positive. That is, Q takes the form*

$$Q(a, u) = \sum_{n \geq 0} u^n P_n(a + 1),$$

where each polynomial P_n has positive coefficients.

Conjecture 2. *the radius of convergence $\rho_Q(a)$ of $Q(a,\cdot)$ is given by*

$$\rho_Q(a) = \begin{cases} \dfrac{1}{(2+\sqrt{2+2a})^2}, & \text{if } a \geq -1/2, \\ \dfrac{-a}{2(a-1)^2}, & \text{if } a \in [-1,-1/2]. \end{cases}$$

Conjecture 3. *The series $Q_u(a,u) = \frac{\partial Q}{\partial u}$ is convergent at $u = \rho_Q(a)$ for $a \geq -1/3$.*

Finally, using the solutions to these two problems, we have computed over 1000 coefficients of each of the series $P(t)$ and $D(t)$. We find that the coefficients of $P(t)$ behave as $\kappa_p \cdot \mu^n \cdot n^\gamma$, where $\mu = 8.281402207\ldots$ and $\gamma \approx -2.473$, while the coefficients of $D(t)$ behave as $\kappa_d \cdot \mu^n \cdot n^{-3/2}$. In particular, we find that the growth rates of these two sequences agree to 10 significant digits, adding credence to the conjecture that $t_p = t_d$.

References

[1] M Albert and M Bousquet-Mélou, *Permutations sortable by two stacks in parallel and quarter-plane walks*, Europ. J. Comb. **43** (2015), 131-164.

[2] D E Knuth, *Fundamental Algorithms, The Art of Computer Programming Vol 1*, Addison-Wesley, Massachusetts (1968).