Coverings, Turán-type Problems, and Thresholds in the Permutation Pattern Lattice

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For $N = 1, 2, \ldots$, the permutation pattern lattice Π_N consists of all the permutations in $S_1 \cup S_2 \ldots \cup S_N$ with $\pi \in S_n < \pi' \in S_{n+1}$ (or π' contains π) if π is contained as a subpattern of π'. We say that a permutation $\pi' \in S_{n+1}$ *covers* a permutation $\pi \in S_n$ if π is contained as a subpattern of π', i.e., if there are n elements of π' that are in the same relative order as π. The key question asked in [1] was the following: what is the minimum number of permutations $\pi' \in S_{n+1}$ necessary to cover all permutations in S_n? In that paper, the authors gave bounds, probabilistic thresholds, and made an unverified claim. In this paper, we extend this research to the general case where π' is a permutation of length $n + k$ for $k \in \mathbb{Z}^+$, and ask what are the minimum number of permutations $\pi' \in S_{n+k}$ necessary to cover all permutations in S_n. We denote this number by $C_{n,n+k}$. Upper and lower bounds on $C_{n,n+k}$ are first proved, featuring a logarithmic gap that we have tried unsuccessfully to remove for $k = 1$ in [1]. We then prove an "iterated logarithm" result for $C^{\lambda}_{n,n+k}$, the minimum number of $(n + k)$-permutations needed to cover all n-permutations $\lambda \geq 2$ times each, exhibiting that substantially fewer permutations are needed for subsequent coverings. Next, for completeness, we include a proof of the unproven result from [1], which states that each permutation π in S_{n+1} covers $(n + 1 - s_\pi)$ n-permutations, where s_π is the number of successions in π. We also show that the number of successions is tightly concentrated in case of a random permutation. Then, we find new results on the problem of "shadowing", or covering backwards, all permutations of length $n + 1$ with permutations of length n. This is a notion that could be classified as being in "Permutation Turán theory" and appears to be a new direction of investigation; by contrast, graph and hypergraph Turán theory is well developed, as seen, e.g., in [3]. Finally, in we prove threshold results of the following type: If we pick $\gg$

$\phi(n)$ or $\ll \phi(n)$ random $(n+k)$-permutations then the probability that we have a successful covering of all n-permutations is asymptotically 1 or 0 as $n \to \infty$ with k fixed. All probabilistic results are obtained via methods such as (Svante) Janson's exponential inequalities studied in [2]. Our work leaves open several open problems, not the least of which is the (asymptotic) enumeration of 2-chains in the permutation pattern lattice (by contrast this number is well-known to be 3^n for the Boolean lattice).

This is joint work with Bill Kay, Emory University; Kathleen Lan, Duke University; Amanda Laubmeier, North Carolina State University; and Ruyue (Julia) Yuan, University of Florida.

References

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