Unimodal inversion sequences and related pattern classes

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June 1, 2017

Mansour and Shattuck [6] have shown that exactly 9 classes of permutations (up to symmetry) avoiding triples of patterns of length 4 have the enumeration sequence A033321 [7] that is the binomial transform of Fine's sequence. Up to symmetry, these classes are $\text{Av}(T_i)$, $i = 1, \ldots, 9$, where

$$T_1 = \{2134, 3124, 4123\}, \qquad T_2 = \{1234, 1324, 1423\}, \qquad T_3 = \{2134, 1324, 1423\}$$
$$T_4 = \{3124, 2314, 2413\}, \qquad T_5 = \{3214, 2314, 2413\}, \qquad T_6 = \{2143, 1243, 1342\}$$
$$T_7 = \{1243, 1342, 1432\}, \qquad T_8 = \{2143, 3142, 4132\}, \qquad T_9 = \{2143, 2413, 3142\}.$$

Four of these classes were separately shown previously to be enumerated by this sequence, while the five others are new. In chronological order of publication, see [1, Lemma 10] for T_2, [4, Example 6.4] for T_9, [9, Theorems 3.4 and 3.5] for T_4, and [3, Section 3.3] for T_8.

The same sequence also enumerates unimodal inversion sequences [8]. An *inversion sequence* (*subexcedant sequence*, a reversal of a *Lehmer code*) is a sequence $e = e_1e_2 \ldots e_n$ ($n \geqslant 0$) such that $e_i \in [0, i-1]$ for all i.

We generalize these findings in two different ways.

- For sets T_1, T_2, T_3, T_8, we find encodings of all permutations by inversion sequences so that the inversion sequences corresponding to the set $\text{Av}(T_i)$ of T_i-avoiders ($i = 1, 2, 3$) are unimodal. This extends our PP 2016 results [5] for T_2 and T_8.

- We use the above encodings of $\text{Av}(T_i)$, $i = 1, 2, 3$, to generalize the Wilf-equivalence of the T_i's to a Wilf-equivalence of families of similarly related sets of patterns of any size, obtained by inflating a certain entry of each permutation in T_i by the same block. We also conjecture similar generalizations for T_i, $i = 4, 5, 6, 7, 8$.

1 Bijections from inversion sequences

Let $\text{SE}_n = \prod_{i=1}^{n} [0, i-1]$ denote the set of all inversion sequences of length n, and let $\text{SE} = \cup_{n \geqslant 0} \text{SE}_n$. Similarly, let UIS_n be the subset of unimodal strings in SE_n, and let UIS be the subset of unimodal strings in SE. Given a permutation π, let (i, j)-*augmentation* of π, denoted $\pi(i, j)$, be the permutation obtained by adding 1 to all letters m of π such that $m \geqslant j$ and inserting letter j to be in position i. Equivalently, $\pi(i,j)$ is the permutation π' such that $\pi'(i) = j$ and the pattern of π' with j deleted is π. We also let

$$\mathrm{Aug}(\pi, i, j) = \{\pi(i,k) \,:\, 1 \leqslant k \leqslant |\pi| + 1,\ k \neq j\}.$$

For example, $123(1,2) = 2134$, $123(1,3) = 3124$, $123(1,4) = 4123$, so $\mathrm{Aug}(123,1,1) = \{2134, 3124, 4123\}$.

Thus,

$$\begin{array}{lll} T_1 = \mathrm{Aug}(123,1,1), & T_4 = \mathrm{Aug}(213,2,2), & T_6 = \mathrm{Aug}(132,2,4), \\ T_2 = \mathrm{Aug}(123,2,1), & T_5 = \mathrm{Aug}(213,2,1), & T_7 = \mathrm{Aug}(132,2,1) \\ T_3 = \mathrm{Aug}(123,2,2), & & T_8 = \mathrm{Aug}(132,1,1). \end{array}$$

We would like to find bijections $f_i : \mathrm{SE}_n \to \mathfrak{S}_n$ such that $f_i(\mathrm{UIS}_n) = \mathrm{Av}(T_i)$. To do that, we first define the bijections g_{132} and g_{123} from the monotone (nondecreasing) strings in SE_n to $\mathrm{Av}(132)$ and $\mathrm{Av}(123)$.

For pattern 132**:** Given $e = e_1 \dots e_n \in \mathrm{SE}_n$, let $\pi^{(0)} = \emptyset$, and for each $i = 1, \dots, n$, let $\pi^{(i)} = \pi^{(i-1)}(1, e_i + 1)$. In other words, at the i-th step, $1 \leqslant i \leqslant n$, insert $e_i + 1$ on the left. Then $\pi = \pi^{(n)} = g_{132}(e)$. In this case, e is just the reversal of the inversion code of π, i.e. e_i is the number of inversions starting from π_{n-i+1}.

For pattern 123**:** Given $e = e_1 \dots e_n \in \mathrm{SE}_n$, let $\pi^{(0)} = \emptyset$, $\pi^{(1)} = 1$, and for each $i = 2, \dots, n$, let

$$\pi^{(i)} = \begin{cases} \pi^{(i-1)}(1, e_i) & \text{if } e_i > e_{i-1}, \\ \pi^{(i-1)}(1, i) & \text{if } e_i = e_{i-1}. \end{cases}$$

Then $\pi = \pi^{(n)} = g_{123}(e)$. In other words, non-right-to-left-maxima of π are the distinct nonzero entries of e.

Now consider all inversion sequences. For each $e \in \mathrm{SE}_n$, let $\mathrm{head}(e)$, the *head* of e, be the longest monotone prefix of e and let $h(e) = |\mathrm{head}(e)|$. In other words, we have $e_1 \leqslant \cdots \leqslant e_{h(e)} > e_{h(e)+1}$. Clearly, when $e \in \mathrm{UIS}_n$, the *tail* of e, i.e. suffix of e starting from $e_{h(e)+1}$, is non-increasing. Then we can define bijections f_1 and f_8 as follows.

For set $T_8 = \mathrm{Aug}(132,1,1)$**:** Let $\pi^{(h(e))} = g_{132}(\mathrm{head}(e))$. Then for each $i = h(e)+1, \dots, n$, let $\pi^{(i)} = \pi^{(i-1)}(1, e_i + 1)$. Then $\pi = \pi^{(n)} = f_8(e)$. This is the simplest case of all, since e is just the inversion code of π as for pattern 132.

For set $T_1 = \mathrm{Aug}(123,1,1)$**:** Let $\pi^{(h(e))} = g_{123}(\mathrm{head}(e))$. Then for each $i = h(e)+1, \dots, n$, let $\pi^{(i)} = \pi^{(i-1)}(1, e_i + 1)$. Then $\pi = \pi^{(n)} = f_1(e)$. This is almost as simple, since the inversion code yields the tail of e, which corresponds to the prefix of π ending on the rightmost "1" in an occurrence of 123 in π.

For sets T_2 and T_3, we need to find the insertion position (that we will call the f-*insertion point*) for each tail entry of e. To find the f-insertion point of a 123-containing permutation σ, which we denote $\text{ins}_f(\sigma)$, find the entry $A(\sigma)$ that is the smallest "1" in an occurrence of pattern 123 in σ, then find the entry $B(\sigma)$ that is the rightmost "2" in an occurrence of pattern 123 in σ that starts with $A(\sigma)$. Then the f-insertion point of σ is immediately to the left of $B(\sigma)$.

Then we can define the bijections f_2 and f_3 as follows.

For set $T_2 = \text{Aug}(123, 2, 1)$**:** Let $\pi^{(h(e))} = g_{123}(\text{head}(e))$. Then for each $i = h(e)+1, \dots, n$, let $\pi' = \pi^{(i-1)}(1, e_i + 1)$, then $\pi^{(i)} = \pi^{(i-1)}(\text{ins}_f(\pi') - 1, e_i + 1)$. In other words, tentatively insert $e_i + 1$ to the left of π, find the f-insertion point of the resulting permutation π', then move the newly-prepended entry $e_i + 1$ of π' to its f-insertion point. Then $\pi = \pi^{(n)} = f_2(e)$.

For set $T_3 = \text{Aug}(123, 2, 2)$**:** Let $\pi^{(h(e))} = g_{123}(\text{head}(e))$. Given a permutation σ, define the map ϕ_σ as follows. Let $b_1 < b_2 < \cdots < b_k$ be the distinct possible values in $\{B(\sigma(1, j)) : 1 \leqslant j \leqslant n\}$. If σ contains 123, then $b_k = B(\sigma)$ and $b_1, \dots, b_{k-1}$ are the left-to-right minima of σ less than $A(\sigma)$ that start a 12 pattern. If σ avoids 123, then all b_j $(1 \leqslant j \leqslant k)$ are the left-to-right minima of σ that start a 12 pattern. Also, set $b_0 = 0$. Let

$$\phi_\sigma(m) = \begin{cases} b_{j-1} + b_j - 1 - m, & \text{if } m \in [b_{j-1}, b_j - 1] \text{ for some } j \leqslant k, \\ m, & \text{if } m \geqslant b_k. \end{cases}$$

Simply put, ϕ_σ maps each interval $[b_{j-1}, b_j - 1]$ onto itself in reverse order and leaves the rest unchanged, and thus is an involution.

Let $d_i = \phi_{\pi^{(i-1)}}(e_i)$. Now for each $i = h(e)+1, \dots, n$, let $\pi' = \pi^{(i-1)}(1, d_i + 1)$, then $\pi^{(i)} = \pi^{(i-1)}(\text{ins}_f(\pi') - 1, d_i + 1)$. In other words, tentatively insert $d_i + 1$ to the left of π, find the f-insertion point of the resulting permutation π', then move the newly-prepended entry $d_i + 1$ of π' to its f-insertion point. Then $\pi = \pi^{(n)} = f_3(e)$.

2 Some generalizations

Notice that each of f_1, f_2, f_3 consists of two parts, with the first part, i.e. the mapping of the head, being the same. Moreover, in the tail mapping part of each of f_2 and f_3, each step uses only the "1" and the "2" of an occurrence of 123 to find the f-insertion point, while in the tail mapping part of f_1, the insertion point is the same at each step. In other words, in the tail mapping part of f_1, f_2, and f_3, no information is needed about any "3" in an occurrence of 123 used to insert the tail entries. This leads to the following generalizations.

Two patterns, σ and τ are called *shape-Wilf-equivalent* (see [2]) if, for every Ferrers board λ, they are Wilf-equivalent on λ. Here we consider Ferrers boards that are complements of the usual Ferrers boards (i.e. are right- and top-justified). Let $\sigma \sim \tau$ denote Wilf-equivalence and $\sigma \overset{s}{\sim} \tau$ denote shape-Wilf-equivalence of σ and τ.

The following result concerns inflations of T_1, T_2, and T_3.

Theorem 2.1 *For any patterns $\sigma \overset{s}{\sim} \tau$, we have*

$$\begin{aligned} &\text{Aug}(123[1,1,\sigma],1,1) \sim \text{Aug}(123[1,1,\sigma],2,1) \sim \text{Aug}(123[1,1,\sigma],2,2) \\ \sim &\text{Aug}(123[1,1,\tau],1,1) \sim \text{Aug}(123[1,1,\tau],2,1) \sim \text{Aug}(123[1,1,\tau],2,2) \end{aligned}$$

We hope to extend it to inflations of T_4 and T_5.

Conjecture 2.2 *The Wilf-equivalence of Theorem 2.1 also includes*

$$\begin{aligned} &\mathrm{Aug}(213[1,1,\sigma],2,1) \sim \mathrm{Aug}(213[1,1,\sigma],2,2) \\ \sim &\mathrm{Aug}(213[1,1,\tau],2,1) \sim \mathrm{Aug}(213[1,1,\tau],2,2) \end{aligned}$$

For T_7 and T_8, it appears that we can inflate two entries and preserve Wilf-equivalence.

Conjecture 2.3 *For any patterns* ρ, σ*,*

$$\mathrm{Aug}(132[1,\rho,\sigma],2,1) \sim \mathrm{Aug}(132[1,\rho,\sigma],1,1)$$

Finally, in one specific case, inflations of all T_i ($i = 1,\dots,8$), appear to be Wilf-equivalent.

Conjecture 2.4 *With* $\rho = 1$ *and* $\sigma = r(\mathrm{id}_m) = m(m-1)\dots 21$ *in the above theorem and conjectures, all patterns in Theorem 2.1 and Conjectures 2.2 and 2.3, as well as* $\mathrm{Aug}(132[1,1,r(\mathrm{id}_m)],2,m+3)$*, are Wilf-equivalent.*

Note that the last of the patterns mentioned in Conjecture 2.4 is an inflation of T_6.

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