# UNIVERSALITY IN SUBSTITUTION-CLOSED PERMUTATION CLASSES

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The aim of this work is to study the asymptotic behavior of a permutation of large size, picked uniformly at random in a *substitution-closed permutation class* given by its (finite or infinite) set of *simple permutations*.

Thanks to their encoding by families of trees, it can be proved that substitution-closed permutation classes (possibly, satisfying additional constraints) share a common behavior. For example, the canonical tree representation of their elements imply that all substitution-closed classes with finitely many simple permutations have an algebraic generating functions. Our work illustrates this universality paradigm in probability theory: we prove that the *biased Brownian separable permuton* is the limiting permuton of many substitution-closed classes (see Theorem 7).

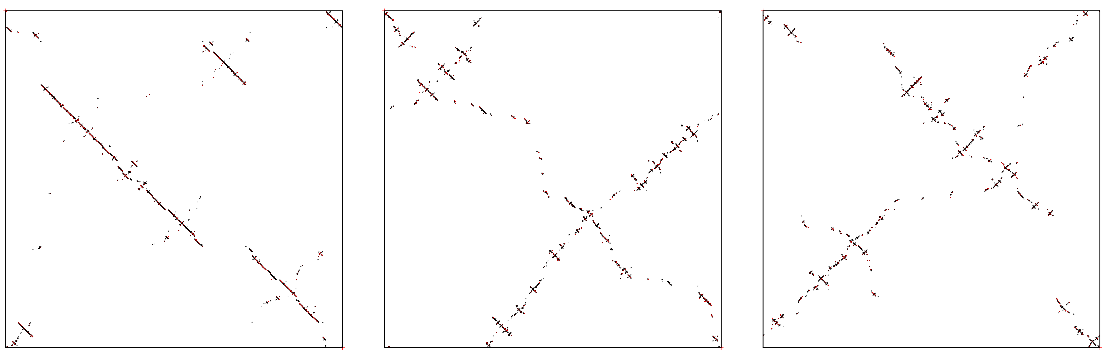

FIGURE 1. Three simulations of the biased Brownian separable permuton $\boldsymbol{\mu}^{(p)}$ (left: $p = 0.2$; middle: $p = 0.45$; right: $p = 0.5$.). By analogy with permutation diagrams, black dots represent the support of the measure $\boldsymbol{\mu}^{(p)}$.

**Context of the work.** One in many ways permutation classes can be studied is by looking at the features of a typical large permutation $\sigma$ in the class. A particularly interesting feature is the frequency of occurrence of a pattern $\pi$, especially when it is considered for all $\pi$ simultaneously. Denote by $\mathrm{occ}(\pi, \sigma)$ the number of occurrences of a pattern $\pi \in \mathfrak{S}_k$ in $\sigma \in \mathfrak{S}_n$ and by $\widetilde{\mathrm{occ}}(\pi, \sigma)$ the *pattern density* of $\pi$ in $\sigma$. More formally

$$\mathrm{occ}(\pi, \sigma) = \mathrm{card}\{I \subset [n] \text{ of cardinality } k \text{ such that } \mathrm{pat}_I(\sigma) = \pi\}$$

$$\widetilde{\mathrm{occ}}(\pi, \sigma) = \frac{\mathrm{occ}(\pi, \sigma)}{\binom{n}{k}} = \mathbb{P}\left(\mathrm{pat}_{\boldsymbol{I}}(\sigma) = \pi\right), \tag{1}$$

where $\boldsymbol{I}$ is randomly and uniformly chosen among the $\binom{n}{k}$ subsets of $[n]$ with $k$ elements. The study of the asymptotics of $\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)$, where $\boldsymbol{\sigma}_n$ is a uniform random permutation of size $n$ in a permutation class $\mathcal{C}$ and $\pi \in \mathfrak{S}$ is a fixed pattern, has been carried out for various classes $\mathcal{C}$ (see [6, 7, 12, 8, 20, 15, 14]).

A parallel line of work to gain understanding on typical large permutations in $\mathcal{C}$ consists in studying the asymptotic shape of the diagram of $\boldsymbol{\sigma}_n$ as $n \to \infty$, possibly after rescaling this diagram so that it fits into a unit square (see [16, 3, 17, 19, 10, 11, 5]).

These two points of view may seem different, but they are in fact tightly bound together. Indeed, it follows from results of [13] that the convergence of pattern densities characterizes the convergence of the diagrams, seen as permutons. This important property was actually the main motivation for the introduction of permutons in [13].

**The permuton viewpoint.** A permuton is a probability measure on the unit square $[0,1]^2$ with both its marginals uniform. Permutons generalize permutation diagrams in the following sense: to every permutation $\sigma \in \mathfrak{S}_n$, we associate the permuton $\mu_\sigma$ obtained by replacing every point $(i,\sigma(i))$ in the diagram of $\sigma$ (normalized to the unit square) by a square of the form $[(i-1)/n, i/n] \times [(\sigma(i)-1)/n, \sigma(i)/n]$, which has mass $1/n$ uniformly distributed.

The weak convergence of measures provides a good notion of convergence for permutons, as discussed in [13]. This allows to define convergent sequences of permutations: we say that $(\sigma_n)$ converges to $\mu$ when $(\mu_{\sigma_n}) \to \mu$ weakly.

By definition, convergence to a permuton encodes the first-order asymptotics of the shape of a sequence of permutations. We can prove that it also encodes the first-order asymptotics of pattern densities:

**Theorem 1.** *A sequence $(\boldsymbol{\sigma}_n)_n$ of random permutations converges in distribution to a random permuton if and only if the sequences of random pattern densities $(\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n))_n$ converge in distribution, jointly for all $\pi \in \mathfrak{S}$.*

*Moreover, for any pattern $\pi$, the limit distribution of the density of $\pi$ can be expressed as a function of the limit permuton.*

Our previous article [4] studies the limit of the class $\mathcal{C} = \mathrm{Av}(2413, 3142)$ of separable permutations, in terms of pattern densities and permutons.

**Theorem 2.** [4] *Let $\boldsymbol{\sigma}_n$ be a uniform random separable permutation of size $n$. There exists a nondeterministic permuton $\boldsymbol{\mu}$, called the Brownian separable permuton, such that $(\mu_{\boldsymbol{\sigma}_n})$ converges in distribution to $\boldsymbol{\mu}$.*

The result of [4] is more precise, and describes the asymptotic joint distribution of the random variables $\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)$ as a measurable functional of a *signed Brownian excursion*. This object is a normalized Brownian excursion whose strict local minima are decorated with an i.i.d. sequence of balanced signs in $\{+,-\}$. The Brownian separable permuton can be directly build from this signed Brownian excursion.

The class of separable permutations is the smallest nontrivial substitution-closed class, as defined in the next section. An important question raised in [4] is: is the Brownian separable permuton universal (in the sense that it describes the limit of a large family of substitution-closed classes)? This works gives a fairly precise (and positive) answer to this question: we will see that in many cases the limit belongs to a one-parameter family of deformations of the Brownian separable permuton: the *biased Brownian separable permuton* $\boldsymbol{\mu}^{(p)}$ of parameter $p \in (0,1)$ is obtained from a biased signed Brownian excursion (defined similarly to the signed Brownian excursion but with each sign having probability $p$ of being a $+$). Simulations of the biased Brownian separable permuton are given in Fig. 1.

**Substitution-closed classes.** The *substitution* $\theta[\pi^{(1)}, \dots, \pi^{(d)}]$ of some permutations $\pi^{(1)}, \dots, \pi^{(d)}$ in a permutation $\theta$ is the permutation obtained by inflating each point $\theta_i$ of $\theta$ by a square containing the diagram of $\pi^{(i)}$ (see Fig. 2). We sometimes refer to $\theta$ as the *skeleton* of the substitution.

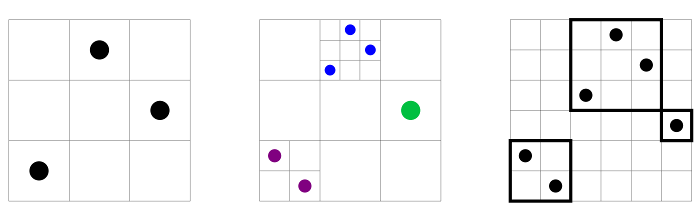

FIGURE 2. Example of substitution: $132[21, 132, 1] = 214653$.

By definition of permutation classes, if $\theta[\pi^{(1)}, \dots, \pi^{(d)}] \in \mathcal{C}$ for some permutation class $\mathcal{C}$, then $\theta, \pi^{(1)}, \dots, \pi^{(d)} \in \mathcal{C}$. The converse is not always true.

**Definition 3.** *A permutation class $\mathcal{C}$ is* substitution-closed *if, for every $\theta, \pi^{(1)}, \dots, \pi^{(d)}$ in $\mathcal{C}$, $\theta[\pi^{(1)}, \dots, \pi^{(d)}] \in \mathcal{C}$.*

The focus of this work is substitution-closed classes. To study such classes it is essential to observe that any permutation has a canonical decomposition using substitutions, which can be encoded in a tree. This decomposition is canonical in the same sense as the decomposition of integers into products of primes. In this analogy, *simple permutations* play the role of prime numbers and the substitution plays the role of the product.

**Theorem 4.** [1] *Every permutation $\sigma$ of size $n \geq 2$ can be uniquely decomposed as either:*

- *$\alpha[\pi^{(1)}, \dots, \pi^{(d)}]$, where $\alpha$ is simple of size $d \geq 4$,*
- *$\oplus[\pi^{(1)}, \dots, \pi^{(d)}]$, where $d \geq 2$ and $\pi^{(1)}, \dots, \pi^{(d)}$ are $\oplus$-indecomposable,*
- *$\ominus[\pi^{(1)}, \dots, \pi^{(d)}]$, where $d \geq 2$ and $\pi^{(1)}, \dots, \pi^{(d)}$ are $\ominus$-indecomposable.*

This decomposition theorem can be applied recursively inside the permutations $\pi^{(i)}$ appearing in the items above, until we reach permutations of size 1. Doing so, a permutation $\sigma$ can be naturally encoded by a rooted planar tree, whose internal nodes are labeled by the skeletons of the substitutions that are considered along the recursive decomposition process, and whose leaves correspond to the elements of $\sigma$. This construction provides a one-to-one correspondence between permutations and *canonical trees* (defined below) that maps the size to the number of leaves.

**Definition 5.** *A* canonical tree *is a rooted planar tree whose internal nodes carry labels satisfying the following constraints.*

- *Internal nodes are labeled by $\oplus, \ominus$, or by a simple permutation.*
- *A node labeled by $\alpha$ has degree $|\alpha|$, nodes labeled by $\oplus$ and $\ominus$ have degree at least 2.*
- *A child of a node labeled by $\oplus$ (resp. $\ominus$) cannot be labeled by $\oplus$ (resp. $\ominus$).*

Let $\mathcal{S}$ be a (finite or infinite) set of simple permutations. We denote by $\langle \mathcal{S} \rangle_n$ the set of permutations of size $n$ whose canonical trees use only nodes $\oplus$, $\ominus$ and $\alpha \in \mathcal{S}$, and we define $\langle \mathcal{S} \rangle = \cup_n \langle \mathcal{S} \rangle_n$. We say that $\mathcal{S}$ is downward-closed if for any $\sigma \in \mathcal{S}$ and any simple pattern $\pi \preccurlyeq \sigma$, it holds that $\pi \in \mathcal{C}$

**Proposition 6.** *Every substitution-closed permutation class $\mathcal{C}$ containing 12 and 21 can be written as $\mathcal{C} = \langle \mathcal{S} \rangle$ for a downward-closed set $\mathcal{S}$ of simple permutations, which is just the set of simple permutations in $\mathcal{C}$.*

*Conversely, for any downward-closed set $\mathcal{S}$ of simple permutations, $\langle \mathcal{S} \rangle$ is a substitution-closed permutation class.*

When a set $\mathcal{S}$ of simple permutations is not downward-closed then $\langle \mathcal{S} \rangle$ is not a permutation class, however the results that we state in this work are also true for this kind of sets of permutations.

**Main result: Universality.** Let $\mathcal{S}$ be a (finite or infinite) set of simple permutations. We are interested in the asymptotic behavior of a uniform permutation $\boldsymbol{\sigma}_n$ in $\langle \mathcal{S} \rangle_n$ which we describe in terms of permutons. Let

$$S(z) = \sum_{\alpha \in \mathcal{S}} z^{|\alpha|}$$

be the generating function of $\mathcal{S}$ and let $R_S \in [0, +\infty]$ be the radius of convergence of $S$.

**Theorem 7** (Main Theorem: the standard case)**.** *Let $\mathcal{S}$ be a set of simple permutations such that*

$$R_S > 0 \quad \text{and} \quad \lim_{\substack{r \to R_S \\ r < R_S}} S'(r) > \frac{2}{(1+R_S)^2} - 1. \tag{H1}$$

*For every $n \geq 1$, let $\boldsymbol{\sigma}_n$ be a uniform permutation in $\langle \mathcal{S} \rangle_n$, and let $\mu_{\boldsymbol{\sigma}_n}$ be the random permuton associated to $\boldsymbol{\sigma}_n$. The sequence $(\mu_{\boldsymbol{\sigma}_n})_n$ tends in distribution in the weak convergence topology to the biased Brownian separable permuton whose parameter $p$ only depends on the quantity of occurrences of the patterns 12 and 21 in the elements of $\mathcal{S}$.*

We call *standard* the case when Condition (H1) is satisfied because there are natural and easy sufficient conditions to ensure this case. Moreover, this case includes all sets $\mathcal{S}$ studied so far in the literature on permutation classes, to our knowledge. Indeed:

- If $S$ is a generating function with radius of convergence $R_S > \sqrt{2} - 1$, (H1) is satisfied. In particular, this covers the cases where there are finitely many simple permutations in the class (then $S$ is a polynomial and $R_S = \infty$), and more generally where the number of simple permutations of size $n$ grows subexponentially (in this case, if there are infinitely many simple permutations, necessarily $R_S = 1$).
- If $S'$ is divergent at $R_S$, (H1) is trivially verified. In particular, this happens when $S$ is a rational generating function, or when $S$ has a square root singularity at $R_S$.

In addition to verifying Condition (H1), we have computed the numerical value of the parameter $p$ for some of sets $\mathcal{S}$ of simple permutations studied in the literature. All the values that we have found are between 0.45 and 0.55, which makes simulations indistinguishable in practice from the unbiased Brownian permuton (see Fig. 1).

Since all the cases studied in the literature (to our knowledge) are covered by the standard case, we may wonder if there exist substitution-closed classes that are not covered by this case. Or are those observations just an artifact, maybe due to the set of simple permutations of a substitution-closed class being easier to compute when Condition (H1) is satisfied? We leave those questions open.

However, when leaving the case of permutation classes, one can easily find sets $\mathcal{S}$ that do not satisfy Condition (H1), and even do that not fit into the universality class of the Brownian permuton. This is discussed in the next section.

**Other results: Beyond universality.** When $R_S > 0$, for the two remaining cases $S'(R_S) < 2/(1+R_S)^2 - 1$ and $S'(R_S) = 2/(1+R_S)^2 - 1$, the asymptotic behavior of $\mu_{\boldsymbol{\sigma}_n}$ is qualitatively different.

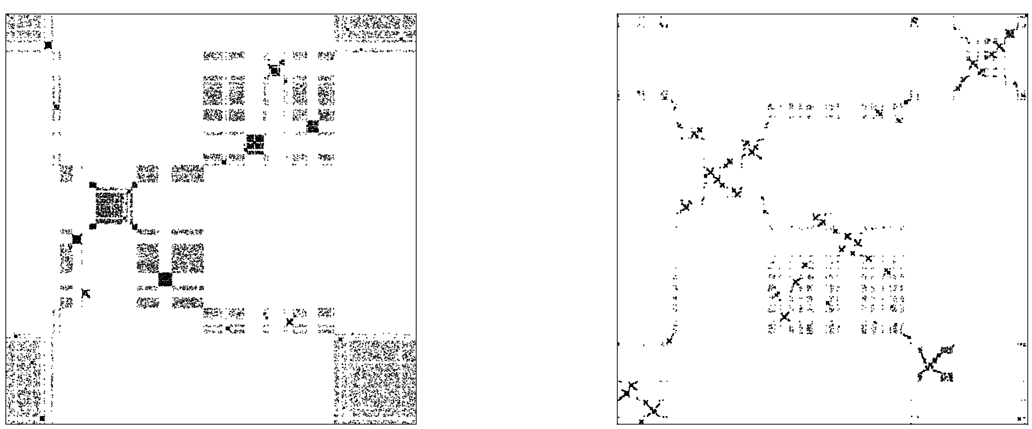

FIGURE 3. Simulations of a 1.1-stable and 1.5-stable permuton.

- **Case** $S'(R_S) < 2/(1+R_S)^2 - 1$. This is a *degenerate* case.
  We can show that, with a small additional assumption which we call $(CS)$, the random permuton $(\mu_{\boldsymbol{\sigma}_n})$ converges. If uniform simple permutations in $\mathcal{S} \cap \mathfrak{S}_n$ have a limit (in the sense of permutons), we show that the limit of permutations in $\langle \mathcal{S} \rangle$ is the same. This explains the terminology "degenerate": all permutations in the class (or set) $\langle \mathcal{S} \rangle$ are close to the simple ones, and the "composite" structure of permutations does not appear in the limit.

- **Case** $S'(R_S) = 2/(1+R_S)^2 - 1$. This *critical* case is more subtle.
  We again need to assume the above mentioned hypothesis $(CS)$. According to the behavior of $S$ near $R_S$, the limiting permuton of $\mu_{\boldsymbol{\sigma}_n}$ can be either the (biased) Brownian separable permuton, or belongs to a new family of *stable* permutons. These stable permutons are defined using the marginals of the random *stable* tree (see [9]), which explains the terminology. Two simulations are presented in Fig. 3.

All cases where we describe the asymptotic behavior of $\mu_{\boldsymbol{\sigma}_n}$ are such that $R_S > 0$. Observe that it is always the case for proper permutation classes (*i.e.*, permutation classes different from $\mathfrak{S}$). Indeed, from the Marcus-Tardos Theorem [18], the number of permutations of size $n$ in a proper class is at most $c^n$, for some constant $c$. For the class $\mathfrak{S}$, we however do have $R_S = 0$, since there are asymptotically $e^{-2}n!(1+\mathcal{O}(1/n))$ simple permutations of size $n$ [2, Theorem 5]. In this case, the permuton $\mu_{\boldsymbol{\sigma}_n}$ associated to a uniform permutation $\boldsymbol{\sigma}_n$ in $\mathfrak{S}$ converges in distribution to the uniform measure on $[0,1]^2$. The situation where $R_S = 0$ may happen as well for sets $\langle \mathcal{S} \rangle$ where $\mathcal{S}$ is not downward-closed, but we leave these cases open.

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