Pattern avoiding permutations modulo pure descents

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Many statistics on permutations have been studied for many years, but two of them appear more frequently in the literature: the number of descents and the number of excedances. Recently [2, 3], two equivalence relations on permutations based on these statistics have been introduced. The main results of these works consist in giving generating functions for the number of equivalence classes for several restricted sets of permutations such as involutions, cycles, derangements, permutations avoiding at most one pattern of length three. Here, we conduct a similar study for an equivalence relation based on the pure descent statistic.

Let π be a permutation in S_n. A *descent* of π is an integer $i \in [n-1]$ such that $\pi_i > \pi_{i+1}$. Whenever there does not exist $j < i$ such that $\pi_{i+1} < \pi_j < \pi_i$, we call it a *pure descent*. Let $D(\pi)$ be the set of pairs (π_i, π_{i+1}) such that i is a pure descent. For instance, if $\pi = 1\ 4\ 2\ 7\ 5\ 3\ 8\ 6$ then $D(\pi) = \{(4,2),(7,5)\}$. In [4, Theorem 1] the authors prove that the number of n-length permutations with k pure descents is given by the signless Stirling number of the first kind (see A132393 in the Sloane's on-line encyclopedia of integer sequences [6]).

We define the following equivalence relation on permutations:

$$\pi \sim \sigma \Longleftrightarrow D(\pi) = D(\sigma).$$

The set of equivalence classes in S_n (resp. in a restricted set $R \subset S_n$) is denoted $S_n^\sim$ (resp. $R^\sim$). For instance, the permutations $\pi = 1\ 4\ 2\ 7\ 5\ 3\ 8\ 6$ and $\sigma = 1\ 7\ 5\ 6\ 4\ 2\ 3\ 8$ belong to the same equivalence class (see Figure 1) because $D(\pi) = D(\sigma) = \{(4,2),(7,5)\}$, and $S_3^\sim$ is constituted of the 5 classes $\{123, 231\}$, $\{132\}$, $\{213\}$, $\{321\}$ and $\{312\}$.

The main goal of our work is to calculate the number of equivalence classes (modulo pure descents) for permutations avoiding at most one pattern of length three. See Table 1 for the results overview.

First of all, we exhibit a one-to-one correspondence between $S_n^\sim$ and the set of noncrossing partitions of $[n]$ prooving that the cardinalities of $S_n^\sim$ for $n \geq 1$ are given by the Catalan numbers (see A000108 in [6]). For the case of permutations avoiding the

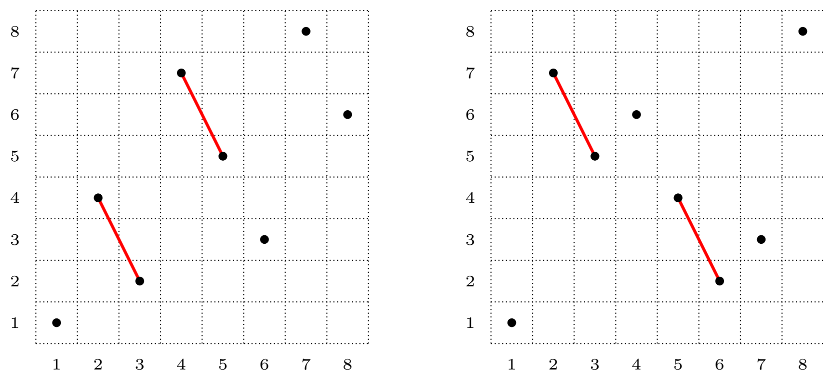

Figure 1: Two permutations $\pi = 1\ 4\ 2\ 7\ 5\ 3\ 8\ 6$ and $\sigma = 1\ 7\ 5\ 6\ 4\ 2\ 3\ 8$ in the same class with $D(\pi) = D(\sigma) = \{(4,2),(7,5)\}$.

pattern 231, any equivalence class contains only one permutation. Also, $S_n(312)^\sim$ and $S_n(321)^\sim$ are enumerated by 2^{n-1} (A011782 in [6]).

Additionally, we describe a bijection between forests of ordered binary trees with n nodes and the set $S_n(231, \underline{51}\bar{4}23)$, giving a new set of pattern avoiding permutations in bijection with the single-source directed animals on the square lattice (see Barcucci and al. [1, 5] and A005773 in [6]). Bivariate generating functions are given for these sets according to various statistics.

Finally, we investigate the equivalence relation on the set $S_n(123)$ of permutations avoiding the pattern 123. We give a constructive bijection between forests of ordered binary trees and the classes in $S_n(123)^\sim$. This proves that the cardinality of $S_n(123)^\sim$ is also given by the sequence A005773 counting the single-source directed animals.

Pattern	Sequence	Sloane	$a_n, 1 \leq n \leq 9$
$\{\}, \{231\}$	Catalan	*A000108*	$1, 2, 5, 14, 42, 132, 429, 1430, 4862$
$\{312\}, \{321\}$	2^{n-1}	*A011782*	$1, 2, 4, 8, 16, 32, 64, 128, 256$
$\{231, \underline{51}\bar{4}23\}$	Directed animals	*A005773*	$1, 2, 5, 13, 35, 96, 267, 750, 2123$
$\{123\}$	Directed animals	*A005773*	$1, 2, 5, 13, 35, 96, 267, 750, 2123$

Table 1: Number of equivalence classes for some restricted sets of pattern avoiding permutations.

Going Further: We obtain experimentally the numbers of classes in $S_n(132)^\sim$ and $S_n(213)^\sim$ for small values of n, $1 \leq n \leq 9$:

- $\#S_n(132)^\sim = 1, 2, 4, 10, 26, 66, 169, 437, 1130, \ldots$;
- $\#S_n(213)^\sim = 1, 2, 4, 9, 22, 56, 146, 388, 1048, \ldots$.

The former sequence does not appear in [6], while the latter seems to be A152225 corresponding to the number of Dyck paths of semilength n with no peaks at height $0 \mod 3$ and no valleys at height $2 \mod 3$. Is it possible to obtain the generating functions for these sets and connect them with Dyck paths?

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