ELMSLEY'S PROBLEM FOR HORSESHOE PERMUTATIONS

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Abstract. We present a general solution to Elmsley's problem for the "horseshoe" permutations. These permutations were first introduced by Butler, Diaconis and Graham in "The mathematics of the flip and horseshoe shuffles" (2016), who presented a solution for horseshoe permutations of size 2^r [1]. Our results extend their results: for horseshoe permutations of any size, we identify the elements that admit a unique Elmsley sequence, and we prove there are exactly two Elmsley sequences otherwise. Moreover, we describe algorithms to compute these sequences.

Elmsley's problem and minimal factorisations in permutation groups

We begin with an everyday description of Elmsley's problem in terms of card shuffling. Suppose we have a deck of cards and that we are only permitted to perform certain kinds of shuffles that we call *allowable shuffles*. Given any card in the deck, we want to find a minimal sequence of allowable shuffles that moves the card to the top of the deck. *Elmsley's problem* consists of describing all such sequences.

By representing a shuffle of a deck of n cards as a permutation acting on the set $\{1, 2, \ldots, n\}$, we can reformulate Elmsley's problem as a minimal factorisation problem for permutation groups. More precisely, let X be a collection of permutations belonging to the symmetric group S_n, and let $\langle X\rangle$ denote the subgroup of S_n generated by X. Each $w \in \langle X\rangle$ can be factored into a product of elements in X, and we let $\ell_X(w)$ denote the minimal number of factors among all such factorisations.

Definition 1. Let $X \subseteq S_n$. For each $c \in \{1, 2, \ldots, n\}$, consider the elements of $\langle X\rangle$ that send c to 1 and that are of minimal length (with respect to ℓ_X):

$$E_c = \{w \in \langle X\rangle : w(c) = 1 \text{ and } \ell_X(w) \leq \ell_X(u) \text{ for all } u \in \langle X\rangle \text{ with } u(c) = 1\}.$$

Elmsley's problem consists of describing the minimal factorisations of these elements (if E_c is nonempty). We call these minimal factorization *Elmsley sequences.*

Elmsley's problem has been solved for various types of shuffles. It was solved for faro shuffles (also called perfect shuffles) by Diaconis and Graham [2]. More recently, Butler, Diaconis and Graham presented a solution to Elmsley's problem for a class of shuffles called *horseshoe shuffles* for decks of cards of size 2^r [1].

Here, we extend their results by presenting a solution to Elmsley's problem for horseshoe shuffles for decks of any size. In addition to describing all minimal factorizations, we provide algorithms to quickly compute the factorizations.

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Elmsley's problem for horseshoe shuffles of a deck of size 2^r.

Horseshoe shuffles are defined on decks with an even number of cards. To perform a horseshoe shuffle, split the deck in half, reverse the order of the cards in the second half, and interlace perfectly the two halves. There are actually two variants of this shuffle depending on the resulting position of the top card: if the top card stays on top, then we have an *out horseshoe shuffle*; otherwise, we have an *in horseshoe shuffle*. See Figure 1. For simplicity, we will refer to these shuffles as *out* and *in*, respectively.

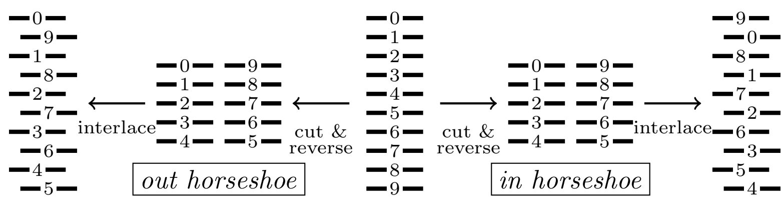

Figure 1. out and in horseshoe shuffles with 10 cards (adapted from [1]).

In [1], Butler, Diaconis and Graham proved the existence of an Elmsley sequence for each card of the deck of size $2n$. For each card in a deck of size 2^r, they also proved that this sequence is unique and they gave an algorithm to find it. They asked for a solution for the general case of a deck of size $2n$.

Due to limited space, we will not give a description of the algorithm here, but we remark that the Elmsley sequence of a card is determined directly from the binary expansion of its position in the deck. For example, to compute the Elmsley sequence of the card in position 7 a deck of size 16, apply the sequence in, in.

Elmsley's problem for horseshoe shuffles of a deck of size $2n$.

We present here an overview of the solution to Elmsley's problem for the *in* and *out* horseshoe shuffles and a deck of size $2n$. The details can be found in [3, 4].

For a deck of $2n$ cards, we will use the factorization $2n = 2^r m$ with r maximal. The problem then breaks up into three parts that will be study separately: "boundary" cards; "middle" cards; and the other cards.

Boundary cards. Given a deck of size $2^r m$ cards with $m > 1$ odd. The *order 1 boundary* is the separation of the deck into two halves that we call *subdecks (of order 1)*. The *order i boundaries* for $2 \leqslant i \leqslant r$ are the separations of the order $i-1$ subdecks into two equals parts. Figure 2 shows an example of boundaries.

A card is a *boundary card* if it is directly above or below a boundary, or if it is the top or bottom card of the deck. We can prove that *out* sends an order i boundary card to an order $i-1$ boundary card; and that *in* sends an order i boundary card to an order $i-1$ boundary card (for $i > 0$). Also, after a shuffle, we know that all cards at order $i-1$ boundaries were all above or all under order i boundaries before the shuffle. To prove this, we use an explicit formula describing the effect of the horseshoe shuffles on the boundary cards. We then get the following result.

Theorem 2. *Every boundary card has a unique Elmsley sequence. If the boundary card is of order i, then its Elmsley sequence is of length $i+1$.*

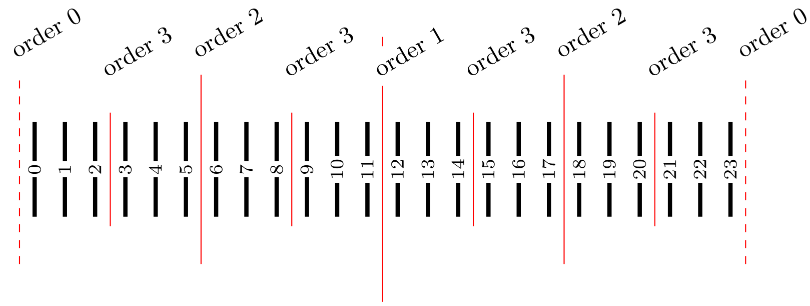

FIGURE 2. Boundaries in a 24 card deck. The boundary cards are those adjacent to the boundaries.

The Elmsley sequence of a boundary card is obtained by applying the Butler, Diaconis and Graham's algorithm in [1] to the deck consisting only of the boundary cards (i.e., the deck obtained by removing the non-boundary cards). More specifically, one uses the binary representation of the position of a card in the smaller deck to determine the (unique) sequence of *in*s and *out*s necessary to move the card to the top of the deck.

For example, the card in position 9 in a deck of size 24 is in position 7 relative to the subdeck of 16 boundary cards. The previous example then tell us that the sequence is in, in.

Middle cards. A *middle card* is a card in the center of one of the smallest subdecks defined by the boundaries. See Figure 3 for two examples.

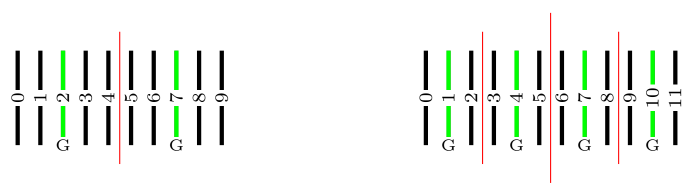

FIGURE 3. In green, middle cards (labelled G) in decks of size 10 and 12 cards.

We prove that both the *in* and *out* horseshoe shuffles move middle cards to the boundary cards with highest possible order, and that all such boundary cards are obtained in this way. In particular, this implies that the Elmsley sequences of all non-boundary cards are not unique.

Theorem 3. *If a card c in a deck of size $2n$ has a unique Elmsley sequence, then either $2n = 2^r$ or $2n = 2^r m$ with $m > 1$ odd and the card c is a boundary card.*

Other cards. We define two types of classes for each card in the deck as follows.

Definition 4. Let x be the position of a card in a deck of $2^r m$ cards with m odd.

- The *type A class* of the card is the set of cards with a position congruent to x mod m. It is denoted $\mathcal{A}_x$.

- The *type B class* of the card is the set of cards with position in
$$\left\{2sm+p,\ (2s+1)m+(m-1-p)\ \middle|\ s\in\{0,1,\dots,2^{k-1}-1\}\right\},$$
where p ($0\leqslant p\leqslant m-1$) is such that the set contains x. It is denoted $\mathcal{B}_x$.

Remark that the type B classes partition the deck. Specifically, a type B class is composed the cards in the odd subdecks that are congruent to p together with the cards in the even subdecks that are congruent to $m-1-p$. In a given subdeck, the latter is symmetric about the middle card with the card congruent to p (see figure 4); more precisely, $\mathcal{B}_p\subseteq\mathcal{A}_p\cup\mathcal{A}_{m-p-1}$.

Note that the type B classes $\mathcal{B}_p$ and $\mathcal{B}_{m-1-p}$ contain cards that are of the same congruence class modulo m. We call two such classes *complementary*. Each $\mathcal{B}_p$ is complementary with exactly one class, namely $\mathcal{B}_{m-1-p}$. Therefore, each type B class has a unique distinct complementary class except for the class of middle cards ($\mathcal{A}_{\frac{m-1}{2}}=\mathcal{B}_{\frac{m-1}{2}}$), which is self-complementary.

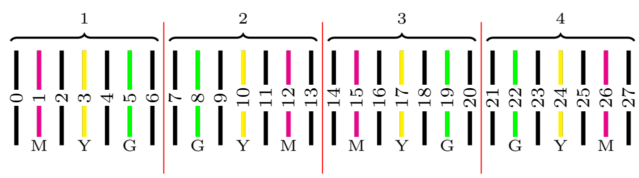

FIGURE 4. The cards in $\mathcal{B}_1$ are in magenta and labelled M; the cards in $\mathcal{B}_5$ are in green and labelled G. Note that $\mathcal{B}_1$ and $\mathcal{B}_5$ are complementary and symmetric about the middle card of the subdecks. The class $\mathcal{B}_3=\mathcal{A}_3$ consists of the middle cards, which are in yellow and labelled Y.

By studying the effect of the horseshoe shuffles on type A and type B classes, we prove the following result.

Proposition 5. *Let U be the union of a type B class with its complementary class. Then the set of cards that can be sent to U by a horseshoe shuffle is a union of the form $\cup_{i\in I}(\mathcal{B}_i\cup\mathcal{B}_{m-1-i})$, with $I\subseteq\{1,2,\dots,m-1\}$. Moreover, $|V|=2|U|$.*

This leads to a proof that there is a unique sequence of minimal length that brings a non-boundary card to a middle card. To achieve this, we begin by constructing the tree T with root $(1,1,1)$ and with branching rule illustrated in Figure 5.

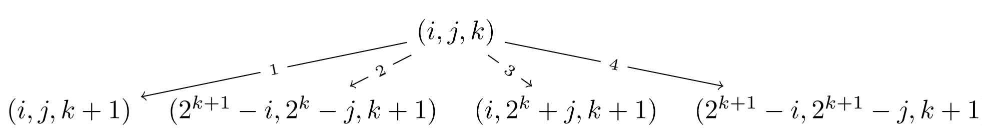

FIGURE 5. Rule for construction of T

Although this tree is independent of m, the trees obtained by applying the function $p_m(i,j,k)=\frac{im-j}{2^k}$ to its vertices allows us to prove that non-boundary cards in positions $tm+p$ and $tm+p+1$ cannot have Elmsley sequences of the same length; and to construct all the Elmsley sequences for each card in the deck. We then have all the tools required to prove the last theorem.

Theorem 6. *For each non-boundary card, the shortest sequence of* in *and* out *horseshoe shuffles that move it to a middle card position is unique. Consequently, all non-boundary cards have exactly two Elmsley sequences.*

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