# ON BOUNCE-FREE RATIONAL LATTICE PATHS

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Abstract. We examine lattice paths from $(0,0)$ to $(\alpha k, \beta k)$ made from only North and East steps, with $\gcd(\alpha, \beta) = 1$. We give generating functions (in terms of known objects) for the number of bounce-free paths (i.e. paths that do not "bounce" off of the line $y = \frac{\beta}{\alpha}x$). Finally, we give examples, some containing possible connections to permutation patterns.

## 1. Definitions

Let $\alpha, \beta \in \mathbb{Z}^+$ with $(\alpha, \beta) = 1$.

**Definition 1.1.** Let $g_{\alpha,\beta}(x) = g(x)$ be the generating function that counts the number of North-East lattice paths from $(0,0)$ to $(\alpha k, \beta k)$, i.e.

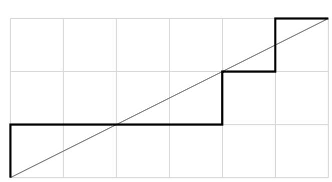
$$g(x) = \sum_{k=1}^{\infty} \binom{(\alpha+\beta)k}{\alpha k} x^k,$$

and let $g^{ab}(x)$ be the generating function that counts the paths that start with an "$a$" step and end with a "$b$" step. (Where $a, b \in \{e, n\}$ with $e =$ East step and $n =$ North step.) Then

$$g^{ee}(x) = \sum_{k=1}^{\infty} \binom{(\alpha+\beta)k-2}{\beta k} x^k, \qquad g^{nn}(x) = \sum_{k=1}^{\infty} \binom{(\alpha+\beta)k-2}{\alpha k} x^k,$$

$$g^{en}(x) = g^{ne}(x) = \sum_{k=1}^{\infty} \binom{(\alpha+\beta)k-2}{\alpha k - 1} x^k.$$

We will consider a path to have a "left bounce" (resp. "right bounce") if the path touches the line $y = \frac{\beta}{\alpha}x$ at a vertex, with the step immediately preceding the vertex being an East step (resp. North step) and the step immediately following the vertex a North step (resp. East step). The figure below contains a path with exactly one right bounce located at vertex $(4,2)$.



Similarly,

**Definition 1.2.** Let $f(x)$ (resp. $f^{ab}(x)$) be the generating function that counts bounce-free paths (resp. bounce-free paths that start with an "$a$" step and end with a "$b$" step). As before, bounces are defined with respect to the line $y = \frac{\beta}{\alpha}x$.

Our goal is to find a generating function for $f(x)$ in terms of $g(x)$ and $g^{en}(x)$ .

## 2. Bounce-free generating functions

**Lemma 2.1.** *For any relatively prime $\alpha, \beta \in \mathbb{Z}^+$,*

$$f^{en}(x) + \frac{f^{ee}(x) f^{nn}(x)}{1 - f^{en}(x)} = \frac{g^{ne}(x)}{g^{ne}(x) + 1}.$$

*Proof.* We claim that both sides of this lemma represent the generating function that counts the paths that begin with a North step, end with an East step, and contain no left bounces. By definition, all bounces happen at a vertex on the line $y = \frac{\beta}{\alpha}x$, and any right bounce (resp. left bounce) can be broken up into two path segments: a path segment that ends with a North step (resp. East step) and a path segment that begins with an East step (resp. North step).

For the left side of this lemma, we start by including the paths with no bounces, counted by $f^{ne}(x) = f^{en}(x)$ (for each path that begins with a North step and ends with an East step, there is a path that begins with an East step and ends with a North step, i.e. the reverse path. For paths with $0 < m$ right bounces, dissect the path into $m + 1$ path segments (each segment starting and ending at a vertex on the line $y = \frac{\beta}{\alpha}x$,

$$p_1^{nn} p_2^{en} p_3^{en} \dots p_{m-1}^{en} p_m^{en} p_{m+1}^{ee},$$

where the $p_i^{ab}$ are bounce-free path segments. These paths can be counted by the generating function

$$f^{nn}(x) \Big( \sum_{i=0}^{\infty} f^{en}(x)^i \Big) f^{ee}(x) = \frac{f^{nn}(x) f^{ee}(x)}{1 - f^{en}(x)}.$$

For the right side of the lemma, we begin with $g^{ne}(x)$, (all paths that start with a North step and end with an East step), and try to remove paths that contain left bounces. Notice that $g^{ne}(x)^{m+1}$ will over-count paths with at least $m$ left bounces, by looking at the path dissection

$$q_1^{ne} q_2^{ne} q_3^{ne} \dots q_{m-1}^{ne} q_m^{ne} q_{m+1}^{ne}.$$

(Note: $q_i^{ne}$ is a path segment that begins with a North step and ends with an East step but has no further restrictions.) By continuously including and excluding paths,

$$g^{ne}(x) - g^{ne}(x)^2 + g^{ne}(x)^3 - \dots = \sum_{i=0}^{\infty} (-1)^i g^{ne}(x)^{i+1} = \frac{g^{ne}(x)}{g^{ne}(x) + 1},$$

we arrive at the result. □

Using similar techniques we can prove the following formulas:

**Lemma 2.2.** *For any relatively prime $\alpha, \beta \in \mathbb{Z}^+$,*

$$\frac{f^{ee}(x)}{1 - f^{en}(x)} = \frac{g^{ee}(x) + g^{en}(x)}{g^{en}(x) + 1} = \frac{\frac{\alpha}{\alpha+\beta} g(x) - g^{en}(x)}{g^{en}(x) + 1}$$

*and*

$$\frac{f^{nn}(x)}{1 - f^{en}(x)} = \frac{g^{nn}(x) + g^{en}(x)}{g^{en}(x) + 1} = \frac{\frac{\beta}{\alpha+\beta} g(x) - g^{en}(x)}{g^{en}(x) + 1}.$$

For convenience, let $\hat{\alpha} = \frac{\alpha}{\alpha+\beta}$ and $\hat{\beta} = \frac{\beta}{\alpha+\beta}$. Combining the three equations from the previous two lemmas, and solving for the three unknowns, we get

**Corollary 2.3.** *For any relatively prime $\alpha, \beta \in \mathbb{Z}^+$, we have*

$$f^{ee}(x) = \frac{g^{en}(x) - \hat{\alpha} g(x)}{(\hat{\alpha} g(x) - g^{en}(x))(\hat{\beta} g(x) - g^{en}(x)) - (1 + g^{en}(x))^2},$$

$$f^{nn}(x) = \frac{g^{en}(x) - \hat{\beta}g(x)}{(\hat{\alpha}g(x) - g^{en}(x))(\hat{\beta}g(x) - g^{en}(x)) - (1 + g^{en}(x))^2},$$
$$f^{en}(x) = f^{ne}(x) = 1 + \frac{1 + g^{en}(x)}{(\hat{\alpha}g(x) - g^{en}(x))(\hat{\beta}g(x) - g^{en}(x)) - (1 + g^{en}(x))^2}.$$

Finally, because $f(x) = f^{ee}(x) + f^{en}(x) + f^{ne}(x) + f^{nn}(x)$, we get the following result.

**Theorem 2.4.** *For any relatively prime $\alpha, \beta \in \mathbb{Z}^+$, the generating function $f(x)$ for the number of bounce-free lattice paths from $(0,0)$ to $(\alpha k, \beta k)$ satisfies*

$$f(x) = 2 + \frac{2 + 4g^{en}(x) - g(x)}{(\hat{\alpha}g(x) - g^{en}(x))(\hat{\beta}g(x) - g^{en}(x)) - (1 + g^{en}(x))^2}$$

*where*

$$g(x) = \sum_{k=1}^{\infty} \binom{(\alpha+\beta)k}{\alpha k} x^k \quad \text{and} \quad g^{en}(x) = \sum_{k=1}^{\infty} \binom{(\alpha+\beta)k - 2}{\alpha k - 1} x^k.$$

**Remark 2.5.** Using the above theorem it is fairly easy to produce generating functions for paths with exactly $\ell$ left bounces and exactly $r$ right bounces, along with many other combinations.

## 3. Examples

**Example 3.1.** For a simple example, the generating function for the number of paths that do not bounce to the right is given by

$$g(x) - \frac{\hat{\alpha}\hat{\beta}g(x)^2}{g^{en}(x) + 1}.$$

If $\alpha \geq 1$ and $\beta = 1$, we have the following two examples.

**Example 3.2.** When $\beta = 1$, the bounce-free generating function reduces to

$$\frac{(2h(x) - 1)^2}{h(x)((1-\alpha)h(x) + 1 + \alpha) - 1}$$

where $h(x)$ is the generating function for the Fuss-Catalan numbers:

$$h(x) = \sum_{k=0}^{\infty} \frac{1}{(\alpha+1)n+1} \binom{(\alpha+1)n+1}{n} x^n.$$

**Example 3.3.** When $\beta = 1$, it is easy to show via Pascal's identity, that

$$g^{ee}(x) = \alpha g^{nn}(x) + (\alpha - 1)g^{en}(x).$$

What is somewhat surprising is the identity

$$f^{ee}(x) = f^{nn}(x) + (\alpha - 1)f^{en}(x).$$

It would be nice to see a combinatorial proof for this equation.

**Example 3.4.** For $\alpha = 2$ and $\beta = 1$, the generating functions for $f^{en}(x)$ and $f^{ee}(x)$ produce integer sequences matching the OEIS entries A259 and A305 (both listed as the number of certain rooted planar maps). Our generating functions produce the simple formulas:

$$A259(n) = \sum_{k=1}^{n} (-1)^{k-1} \binom{3n}{n-k} \frac{k}{n} F_{k-2},$$

and
$$A305(n) = \sum_{k=1}^{n} (-1)^k \binom{3n}{n-k} \frac{k}{n} F_{k-3},$$
where the sequence $\{F_n\}$ is the Fibonacci sequence (with the typical convention that $F_{-2} = -1$, $F_{-1} = 1$ and $F_0 = 1$.)

Finally,

**Example 3.5.** When $\alpha = 1$ and $\beta = 1$, the generating function for the number of paths with a total of $m$ bounces (any combination of left or right bounces) is $2(c(x) - 1)^{m+1}$ where $c(x)$ is the generating function for the Catalan numbers. With $m = 2$ we can see a potential connection between paths from $(0, 0)$ to $(n, n)$ that bounce exactly twice (normalized by having the first bounce be a right bounce), and the number of permutations of $[n]$ with exactly 1 increasing subsequence of length 3 (OEIS entry A3517). A combinatorial proof of this may lead to more connections between permutation patterns and bounce paths.

## 4. References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