Gray codes for restricted ascent sequences

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An *ascent sequence* is a sequence $x_1x_2\cdots x_n$ of nonnegative integers such that $x_1 = 0$ and

$$x_i \leq \mathbf{asc}(x_1x_2\cdots x_{i-1}) + 1 \tag{1}$$

for $2 \leq i \leq n$, where $\mathbf{asc}(x_1x_2\cdots x_k)$ is the number of *ascents* in the sequence $x_1x_2\cdots x_k$, that is, the number of $1 \leq j$ such that $x_j < x_{j+1}$. For example, 01201014216 is an ascent sequence of length 11 and 0120106 is not an ascent sequence, because the 7th position is greater than $\mathbf{asc}(012010) + 1 = 4$. The number of ascent sequences of length n is enumerated by Fishburn number, see A022493 in OEIS. For a nonnegative integer sequence $x = x_1x_2\cdots x_n$, the *reduction* of x is the the sequence obtained by replacing the ith smallest digits of x with $i-1$ and we denote it by $\mathbf{red}(x)$. Also a nonnegative sequence x is called *reduced* if $x = \mathbf{red}(x)$. For example, $\mathbf{red}(32279995) = 10034442$ and 0101232 is a reduced sequence and 01475 is not. An ascent sequence is not always reduced.

For a reduced sequence $p_1p_2\cdots p_k$, an ascent sequence $x = x_1x_2\cdots x_n$ is called a $p = p_1p_2\cdots p_k$-*avoiding ascent sequence* if $\mathbf{red}(x_{i_1}x_{i_2}\cdots x_{i_k}) \neq p_1p_2\cdots p_k$, for all $1 \leq i_1 < i_2 < \cdots i_k \leq n$. This definition is analogous to that of pattern avoiding permutations. Enumerative results and weight enumerations about pattern avoidance in ascent sequences is found at [1] ant its references.

Combinatorial Gray code is a method to generate objects in a combinatorial class so that the successive objects differ in some pre-specified, usually small way. There are many ad-hoc techniques to construct Gray codes for given combinatorial objects. Many of them are listed at Savage's paper [3]. Recently, more general techniques are proposed and many Gray codes are obtained, see [4] and its references.

In this presentation, we focus on the Gray codes for pattern avoiding ascent sequences. Let $\mathcal{A}(n)$ be the set of ascent sequences of length n, where $n \in \mathbb{N}$. Sabri and Vajnovszki gave a Gray code for $\mathcal{A}(n)$ [2]. Define $\mathcal{A}_p(n)$ to be the ascent sequences of length n avoiding a pattern p.

We define two distances and denote them by d_{Ham} and d_{Str}. For two ascent sequence of length n, say $x = x_1x_2\cdots x_n$ and $y = y_1y_2\cdots y_n$, define

$$d_{Ham}(x, y) := |\{i | x_i \neq y_i, 1 \leq i \leq n\}| \tag{2}$$

and

$$d_{Str}(x, y) := \sum_{1 \leq i \leq n} |x_i - y_i|. \tag{3}$$

We call d_{Ham} (resp. d_{Str}) *Hamming distance* (resp. *strong distance*). Hamming distance is the number of positions where the corresponding elements are different. The strong distance is the total sum of the differences of elements for all positions. Notice that strong distance is stronger than

Hamming distance, that is, the strong distance of two given ascent sequences is equal or greater than the Hamming distance of them. For a subset S of the ascent sequences of fixed length, we say that S has a Gray code of Hamming (resp. Strong) distance k if there is a listing of S such that two successive sequences differ at most k in the Hamming (resp. strong) distance.

Also we say that two ascent sequences $x = x_1x_2 \cdots x_n$ and $y = y_1y_2 \cdots y_k$ differ by an adjacent transposition if $x_i = y_i$ for $i \notin \{k, k+1\}$ and $x_k = y_{k+1}, x_{k+1} = y_k$ for some $1 \leq k \leq n-1$.

From general approach, we obtain the following two results.

Theorem 1. *For a reduced sequence $p_1p_2 \cdots p_k$ with $\mathbf{max}\{p_1, p_2, \cdots p_k\} \geq 2$, $A_{p_1p_2\cdots p_k}(n)$ has a Gray code of Hamming distance one.*

Theorem 2. *For a reduced sequence $p_1p_2 \cdots p_k$ with $p_k \geq \mathbf{max}\{p_1, p_2, \cdots p_{k-1}\} + 1$ and $a_k \geq 2$, $A_{p_1p_2\cdots p_k}(n)$ has a Gray code of strong distance at most two.*

The author's preprint [5] yields the following result.

Theorem 3. *$A_{010}(n)$ and $A_{001}(n)$ has a Gray code of strong distance at most two. Moreover, distance two jumps do not appear consecutively.*

Also we can prove the following by ad-hoc argument.

Theorem 4. *$A_{011}(n)$ has a Gray code such that two successive sequences differ by an adjacent transposition or one position.*

References

[1] A. Baxter, L. Pudwell, Ascent sequences avoiding pairs of patterns, Electronic Journal of Combinatorics 22.1, P1.58, 2015

[2] A. Sabri, V. Vajnovszki, Reflected Gray code based orders on some restricted growth sequences, The Computer Journal 58(5), 1099-1111, 2015.

[3] C. Savage. A Survey of Combinatorial Gray Codes, SIAM Review, v.39 n.4, p.605-629, Dec, 1997.

[4] A. Sabri, Gray codes and efficient exhaustive generation for several classes of restricted words, https://tel.archives-ouvertes.fr/tel-01203339/document.

[5] M. Tomie, Two listing of binary words and Gray codes for 132-312 avoiding permutations, arXiv:1612.03620.