Pattern Avoiding Generalized Alternating Permutations

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1 Introduction

In one of the best known results in the theory of pattern avoiding permutations, Knuth proved in 1968 that the number of permutations in S_n avoiding the pattern 123 is given by the nth Catalan number, C_n, which is also counted by the number of standard Young tableaux of shape $< 2^n >$. In 2011, Lewis [2] extended this idea to alternating permutations, i.e. permutations $\pi = \pi_1\pi_2\cdots\pi_n$ such that $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \cdots$. Lewis proved that the number of alternating permutations of length $2n$ avoiding the pattern 1234 is counted by the number of standard Young tableaux of shape $< 3^n >$. Lewis generalized further to the set $L_{n,k}$ of permutations $\pi = \pi_{11}\pi_{12}\pi_{13}\cdots\pi_{1k}\pi_{21}\pi_{22}\cdots\pi_{2k}\cdots\pi_{n1}\pi_{n2}\cdots\pi_{nk}$ of length nk such that $\pi_{i1} < \pi_{i2} < \cdots < \pi_{ik}$ for $1 \leq i \leq n$ by proving that the number of permutations in $L_{n,k}(123\cdots k(k+1)(k+2))$ is counted by the number of standard Young tableaux of shape $< (k+1)^n >$.

In 2017, Mei and Wang [1] further extended Lewis' results to permutations in $L_{n,k,I}(123\cdots k(k+1)(k+2))$ for an index set $I \subseteq [n]$ and proved that these permutations were counted by standard Young tableaux of shape $< (k+1)^n >$ independent of index set I. In their paper containing this result, Mei and Wang pose the question about finding a direct bijection between the pattern avoiding permutations in $L_{n,k,I}$ for different index sets I that does not rely on the RSK correspondence and the standard Young tableaux. This talk will answer that open question by giving such a bijection.

2 Background

We consider the set $L_{n,k,I}(123\cdots(k+1)(k+2)$ of pattern avoiding permutations studied by Mei and Wang in [1]. For ease of notation in this talk, we will let $\mathcal{L}(n,k,I)$ mean $L_{n,k,I}(123\cdots(k+1)(k+2)$.

Definition 2.1. Let n,k be positive integers and let I be an index set $I \subseteq [n]$. Then $\mathcal{L}(n,k,I)$ is the set of permutations $\sigma \in S_{nk+|I|}$ such that

$$\sigma = \sigma_{11}\sigma_{12}\cdots\sigma_{1j_1}\sigma_{21}\cdots\sigma_{2j_2}\cdots\sigma_{n1}\cdots\sigma_{nj_n}$$

where

(C1) $j_i = k+1$ if $i \in I$ and $j_i = k$ otherwise

(C2) $\sigma_{i1} < \sigma_{i2} < \cdots < \sigma_{ij_i}$ for all $1 \leq i \leq n$

(C3) σ avoids the pattern $123\cdots(k+1)(k+2)$.

Example 2.2. Consider the set $\mathcal{L}(5,2,\{1,3\})$. By Definition 2.1, the set contains (1234)-avoiding permutations in S_{12} that can be split into 5 **blocks**, with blocks one and three of length three and the remaining three blocks of length two. By definition, the elements in each block are increasing from left to right. We will represent such a permutation by indicating the blocks using underbrackets. One such element $\sigma \in \mathcal{L}(5,2,\{1,3\})$ is given by:

$$\sigma = \underbrace{7\ 10\ 12}\,\underbrace{6\ 9}\,\underbrace{1\ 5\ 11}\,\underbrace{3\ 8}\,\underbrace{2\ 4}.$$

In 2011, Lewis [2] gave a thorough study of $\mathcal{L}(n,k,\emptyset)$ and the relation of this set of pattern avoiding permutation to standard Young tableaux of shape $< (k+1)^n >$, giving a bijection between them [Theorem 4.1]. Mei and Wang [1] extend Lewis's bijection to a bijection between $\mathcal{L}(n,k,I)$ and the set of standard Young tableaux of shape $< (k+1)^n >$ for any set $I \subseteq [n]$. An interesting Corollary is that $|\mathcal{L}(n,k,I)|$ is independent of the choice of I, thus raising the question of finding a bijection between $|\mathcal{L}(n,k,I)|$ and $|\mathcal{L}(n,k,I')|$ for any two index sets I and I' in $[n]$. Mei and Wang [1] give a bijection (Theorem 2.3) between $|\mathcal{L}(n,k,I)|$ and the set of standard Young tableaux of shape $< (k+1)^n >$ which utilizes the RSK correspondence. Composing this bijection and its inverse can give a mapping between $|\mathcal{L}(n,k,I)|$ and $|\mathcal{L}(n,k,I')|$, however this map is not intuitive and is a bit technical in the details. Mei and Wang posed the open question of finding a direct bijection between these sets that didn't make use of the standard Young tableaux. We answer this question below, first for index sets where $|I| = |I'|$ and then in general.

3 Main Results

We first give a bijection between $\mathcal{L}(n,k,I)$ and $\mathcal{L}(n,k,I')$ in the case where $|I| = 1 = |I'|$ and then we generalize to any size I and I'.

In [1], the authors give a bijection between $\mathcal{L}(n,1,\{1\})$ and $\mathcal{L}(n,1,\emptyset)$. Below we extend this idea to any index set of size one.

Proposition 3.1. *There is a bijection between $\mathcal{L}(n,1,\{1\})$ and $\mathcal{L}(n,1,\{k\})$*

Proof. Let $\sigma = \sigma_{11}\sigma_{12}\sigma_{21}\sigma_{31}\cdots\sigma_{n1} \in \mathcal{L}(n,1,\{1,\})$. First note that σ is (123)-avoiding $\sigma_{11} < \sigma_{12}$ by definition. We then define the bijection iteratively. At step 1, to create $\sigma^2 \in \mathcal{L}(n,1,\{2,\})$ compare σ_{11} to σ_{21}.

- If $\sigma_{11} > \sigma_{21}$, swap σ_{12} and σ_{21}. I.e., move σ_{12} to the position immediately to the **right** of σ_{21}. Since σ is (123)-avoiding, we know that $\sigma_{21} < \sigma_{12}$, thus the resulting permutation is in $\mathcal{L}(n,1,\{2\})$.

- If $\sigma_{11} < \sigma_{21}$, swap $\sigma 11$ and σ_{12}. I.e., move σ_{11} to the position immediately to the **left** of σ_{21}. Since $\sigma_{11} < \sigma_{21}$, the resulting permutation is in $\mathcal{L}(n,1,\{2\})$.

Call the resulting permutation σ^2.

In general, if σ^i is a permutation in $\mathcal{L}(n,1,\{i\})$, given as

$$\sigma^i = \sigma^i_{11}\sigma^i_{21}\cdots\sigma^i_{(i-1)1}\sigma^i_{i1}\sigma^i_{i2}\sigma^i_{(i+1)1}\cdots\sigma^i_{n1},$$

then to create a permutation in $\mathcal{L}(n,1,\{i+1\})$ do the following:

- If $\sigma^i_{i1} > \sigma^i_{(i+1)1}$, swap σ^i_{i2} and $\sigma^i_{(i+1)1}$. I.e., move σ^i_{i2} to the position immediately to the **right** of $\sigma^i_{(i+1)1}$. Since σ^i is (123)-avoiding, we know that $\sigma^i_{(i+1)1} < \sigma^i_{i2}$, thus the resulting permutation is in $\mathcal{L}(n,1,\{i+1\})$.

- If $\sigma^i_{i1} < \sigma^i_{(i+1)1}$, swap σ^i_{i1} and σ^i_{i2}. I.e., move σ^i_{i1} to the position immediately to the **left** of $\sigma^i_{(i+1)1}$. Since $\sigma^i_{i1} < \sigma^i_{(i+1)1}$, the resulting permutation is in $\mathcal{L}(n,1,\{i+1\})$.

□

Example 3.2. Let $n = 9, k = 1$ and $I = \{4\}$. Consider the permutation $\sigma \in \mathcal{L}(9,1,\{4\})$ given by:

$$\sigma = \underbrace{9}\,\underbrace{8}\,\underbrace{7}\,\underbrace{5\ 10}\,\underbrace{4}\,\underbrace{2}\,\underbrace{6}\,\underbrace{3}$$

Since $\sigma_{41} = 5 > 4 = \sigma_{51}$, then move $\sigma_{42} = 10$ to the right of $\sigma_{51} = 4$, resulting in the permutation $\sigma' \in \mathcal{L}(9, 1, \{5\})$ given by:

$$\sigma' = \underbracket{9}\,\underbracket{8}\,\underbracket{7}\,\underbracket{5}\,\underbracket{4\ 10}\,\underbracket{2}\,\underbracket{6}\,\underbracket{3}$$

Now consider the more general case when k is larger than 1.

Proposition 3.3. *There is a bijection between $\mathcal{L}(n, k, \{i\})$ and $\mathcal{L}(n, k, \{i + 1\})$*

In general, if σ^i is a permutation in $\mathcal{L}(n, k, \{i\})$ let's consider the ith and the $(i + 1)$st blocks, which look like:

$$\sigma_{i1}\sigma_{i2} \cdots \sigma_{ik}\sigma_{i(k+1)}\sigma_{(i+1)1}\sigma_{(i+1)2} \cdots \sigma_{(i+1)k}.$$

To create a permutation in $\mathcal{L}(n, k, \{i + 1\})$ do the following:

- If $\sigma_{i1} < \sigma_{(i+1)1}$, move σ_{i1} to the position immediately to the **left** of $\sigma_{(i+1)1}$.
- If $\sigma_{i1} > \sigma_{(i+1)1}$, compare σ_{i2} and $\sigma(i + 1)2$. If $\sigma_{i2} < \sigma_{(i+1)2}$ the move σ_{i2} to the position immediately to the left of $\sigma_{(i+1)2}$. Since the original permutation avoids $(123 \cdots (k + 1)(k + 2)$, $\sigma_{(i+1)1} < \sigma_{i2}$, thus the resulting permutation is in $\mathcal{L}(n, 1, \{i + 1\})$.
- Continue this process inductively until some element of the ith block is moved to the $(i+1)$st block or $\sigma_{il} > \sigma_{(i+1)l}$ for all $1 \leq l \leq k$. In this case, move $\sigma_{i(k+1)}$ to the position immediately to the right of $\sigma_{(i+1)k}$.

Theorem 3.4. *For an I and I' for which $|I| = |I'|$, there is a bijection between $\mathcal{L}(n, k, I)$ and $\mathcal{L}(n, k, I')$.*

Proof. The proof follows directly by utilizing the bijection in Proposition 3.3 both directly and inversely to change I into I'. □

Theorem 3.5. *For any I and I', there is a bijection between $\mathcal{L}(n, k, I)$ and $\mathcal{L}(n, k, I')$.*

Proof. The proof follows by utilizing Theorem 3.4 and by extending the bijection given by Mei and Wang [1] from $\mathcal{L}(n, 1, \{1\})$ and $\mathcal{L}(n, 1, \emptyset)$ to $\mathcal{L}(n, k, \{1\})$ and $\mathcal{L}(n, k, \emptyset)$. □

References

[1] Zhousheng Mei, Suijie Wang, *Pattern Avoidance and Young Tableaux* Electronic Journal of Combinatorics **24(1)** (2017), 6 pps.

[2] Joel Brewster Lewis, *Pattern Avoidance for Alternating Permutations and Young Tableaux* J. Combin. Theory Ser. A **110** (2011), 1436-1450.