On super-strong Wilf equivalence classes of permutations

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In this work we investigate the notion of super-strong Wilf equivalence, providing two characterizations in terms of distances between letters within a permutation, or via a construction of a particular two-colored binary tree.

Let $\mathbb{P}$ be the set of positive integers. For a word $w \in \mathbb{P}^*$, let $|w|$ be its length and $\|w\|$ be its height, i.e. the sum of its letters. The *weight* of w is the monomial $t^{|w|}x^{\|w\|}$. Suppose $u, w \in \mathbb{P}^*$. An embedding of u in w is a factor v of w such that $|v| = |u| = k$ and $u_i \leq v_i$ for all $i \in [k]$. If the first element of v is the j-th element of w then the index j is called an *embedding index* of u into w. The *embedding set* $Em(u, w)$ is the set of all embedding indices of u into w.

There are three basic types of Wilf equivalence related to our work. Two words $u, v \in \mathbb{P}^*$ are called *Wilf equivalent* if they share the same weight generating function, as defined in [1]. They are called *super-strong Wilf equivalent* (resp. *strong Wilf equivalent*) if there exists a weight-preserving bijection f such that $Em(u, w) = Em(v, f(w))$ (resp. $|Em(u, w)| = |Em(v, f(w))|$), for all $w \in \mathbb{P}^*$.

A well-known negative criterion for super-strong Wilf equivalence is related to the notion of *minimal cluster* for a certain embedding index set E. Given a word u and a set E, the *minimal cluster* $m(u, E)$ of u with embedding set E is the unique word w such that $Em(u, w) = E$ and none of the entries of w can be decreased without altering the set of its factors.

An open problem in this area is the *Rearrangement Conjecture* stating that two Wilf equivalent words have to be rearrangements of one another. This has been settled in the case of strong Wilf equivalent words in [2], using minimal clusters. In this direction, our first result is the *Minimal Cluster Rearrangement Theorem* (MCRT).

Theorem 1. (MCRT) *Let $u_1, u_2 \in \mathbb{P}^*$. Then $u_1 \sim_{ss} u_2$ if and only if the minimal clusters $m(u_1, E)$ and $m(u_2, E)$ are rearrangements of one another, for every embedding index set E.*

Let $\overline{j}$ be the shift of an embedding set E by $(j-1)$ positions to the right. An arithmetic interpretation of MCRT is the following result.

Proposition 1. *Let $u, v \in \mathcal{S}_n$ and $s = u^{-1}, t = v^{-1}$. Then $u \sim_{ss} v$ if and only if*

$$|\overline{s_i} \cap (\bigcup_{j=i+1}^{n} \overline{s_j})| = |\overline{t_i} \cap (\bigcup_{j=i+1}^{n} \overline{t_j})|, \tag{1}$$

for each $i \in [n-1]$ and every embedding set E.

The previous result implies preservation of distances under super-strong Wilf equivalence. This leads naturally to the notion of *cross equivalence*. We say that u, $v \in \mathcal{S}_n$ are cross equivalent if for each letter i, $i^+(u) = i^+(v)$, where $i^+(u)$ denotes the multiset of distances of i from letters greater than i in u.

Using the Inclusion-Exclusion Principle we can simplify Proposition 1. For this purpose we define the notion of *consecutive differences*. Given a permutation u, its inverse $s = u^{-1}$ and a letter i, the vector of consecutive differences $\Delta_i(s)$ for $i \in [2, n-1]$, contains the distances between letters in u that are greater than or equal to i as they appear sequentially in u from left to right. Our main result is the following concrete characterization of super-strong Wilf equivalence.

Theorem 2. *Let $u, v \in \mathcal{S}_n$ and $s = u^{-1}, t = v^{-1}$. Then $u \sim_{ss} v$ if and only if $\Delta_i(s) = \Delta_i(t)$, for each $i \in [2, n-1]$.*

In the sequel, we define a binary tree $T^n(u)$ that helps us visualize the cross equivalence class of a given permutation u as the set of leaves of $T^n(u)$. The crucial point here is that the cardinality of the latter is always a power of 2. In order to partition this set into super-strong Wilf equivalence classes, we define a labeling on those levels of $T^n(u)$ that contain vertices which have two children. This labeling distinguishes between "good" vertices which preserve symmetry (labeled 0), and "bad" ones which destroy it (labeled 1). This is in accordance to the sequence of differences $\Delta_i(s)$ for $i \in [2, n-1]$ and implies our final result.

Theorem 3. *Let $u \in \mathcal{S}_n$ and let k, l be the number of levels in $T^n(u)$ labeled 0 and 1, respectively. Then:*

- *The number of words in each super-strong Wilf equivalence class in $T^n(u)$ is equal to 2^k.*
- *The class $[u]_+$ is partioned into 2^l distinct super-strong Wilf equivalence classes.*

This theorem verifies, in the case of super-strong Wilf equivalence, the conjecture stated in [1] that the cardinality of each Wilf equivalence class of a permutation is a power of 2.

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References

[1] S. Kitaev, J. Liese, J. Remmel, B. E. Sagan, *Rationality, irrationality and Wilf equivalence in generalized factor order*, The Electronic Journal of Combinatorics 16(2), 2009.

[2] J. Pantone, V. Vatter, *On the Rearrangement Conjecture for generalized factor order over* $\mathbb{P}$, in 26th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2014). Discrete Math. Theor. Comput. Sci. Proc., AT. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2014, pp. 217-228.