Quadrant marked mesh patterns in 123-avoiding permutations

Dun Qiu and Jeffrey B. Remmel, Department of Mathematics
University of California, San Diego
La Jolla, CA 92093-0112

1 Introduction

Given a sequence $w = w_1 \dots w_n$ of distinct integers, let red$[w]$ be the permutation found by replacing the i-th largest integer that appears in σ by i. For example, if $\sigma = 2754$, then red$[\sigma] = 1432$. Given a permutation $\tau = \tau_1 \dots \tau_j$ in the symmetric group S_j, we say that the pattern τ *occurs* in $\sigma = \sigma_1 \dots \sigma_n \in \mathcal{S}_n$ provided there exists $1 \leq i_1 < \cdots < i_j \leq n$ such that red$[\sigma_{i_1} \dots \sigma_{i_j}] = \tau$. We say that a permutation σ *avoids* the pattern τ if τ does not occur in σ. Let $\mathcal{S}_n(\tau)$ denote the set of permutations in $\mathcal{S}_n$ which avoid τ.

The main goal of this paper is to study the distribution of quadrant marked mesh patterns in 123-avoiding permutations. The notion of mesh patterns was introduced by Brändén and Claesson [1] to provide explicit expansions for certain permutation statistics as, possibly infinite, linear combinations of (classical) permutation patterns. Kitaev and Remmel initiated the systematic study of distribution of quadrant marked mesh patterns on permutations in [3]. This study was extended to 132-avoiding permutations by Kitaev, Remmel and Tiefenbruck in [4, 5, 6].

Let $\sigma = \sigma_1 \dots \sigma_n$ be a permutation written in one-line notation. Then we will consider the graph of σ, $G(\sigma)$, to be the set of points (i, σ_i) for $i = 1, \dots, n$. For example, the graph of the permutation $\sigma = 471569283$ is pictured in Figure 1. Then if we draw a coordinate system centered at a point (i, σ_i), we will be interested in the points that lie in the four quadrants I, II, III, and IV of that coordinate system as pictured in Figure 1. For any $a, b, c, d \in \mathbb{N}$ where $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of natural numbers and any $\sigma = \sigma_1 \dots \sigma_n \in \mathcal{S}_n$, we say that σ_i matches the quadrant marked mesh pattern MMP(a, b, c, d) in σ if, in $G(\sigma)$ relative to the coordinate system which has the point (i, σ_i) as its origin, there are at least a points in quadrant I, at least b points in quadrant II, at least c points in quadrant III, and at least d points in quadrant IV. For example, if $\sigma = 471569283$, the point $\sigma_4 = 5$ matches the marked mesh pattern MMP$(2, 1, 2, 1)$ since, in $G(\sigma)$ relative to the coordinate system with the origin at $(4, 5)$, there are 3 points in quadrant I, 1 point in quadrant II, 2 points in quadrant III, and 2 points in quadrant IV. Note that if a coordinate in MMP(a, b, c, d) is 0, then there is no condition imposed on the points in the corresponding quadrant.

In addition, we shall consider the patterns MMP(a, b, c, d) where $a, b, c, d \in \mathbb{N} \cup \{\emptyset\}$. Here when a coordinate of MMP(a, b, c, d) is the empty set, then for σ_i to match MMP(a, b, c, d) in $\sigma = \sigma_1 \dots \sigma_n \in \mathcal{S}_n$, it must be the case that there are no points in $G(\sigma)$ relative to the coordinate system with the origin at (i, σ_i) in the corresponding quadrant. For example, if $\sigma = 471569283$, the point $\sigma_3 = 1$ matches the marked mesh pattern MMP$(4, 2, \emptyset, \emptyset)$ since, in $G(\sigma)$ relative to the coordinate system with the origin at $(3, 1)$, there are 6 points in quadrant I, 2 points in quadrant II, no points in quadrants III and IV. We let mmp$^{(a,b,c,d)}(\sigma)$ denote the number of i such that σ_i matches MMP(a, b, c, d) in σ.

For any permutation τ, we let

$$Q_{\tau}^{(a,b,c,d)}(t, x) = 1 + \sum_{n \geq 1} t^n Q_{n,\tau}^{(a,b,c,d)}(x)$$

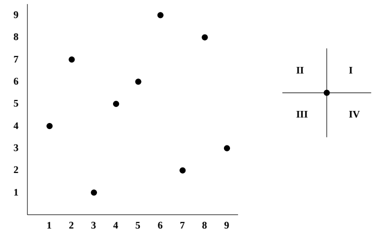

Figure 1: The graph of $\sigma = 471569283$.

where for any $a, b, c, d \in \{\emptyset\} \cup \mathbb{N}$, $Q_{n,\tau}^{(a,b,c,d)}(x) = \sum_{\sigma \in \mathcal{S}_n(\tau)} x^{\mathrm{mmp}^{(a,b,c,d)}(\sigma)}$. For any a, b, c, d, we will write $Q_{n,\tau}^{(a,b,c,d)}(x)|_{x^k}$ for the coefficient of x^k in $Q_{n,\tau}^{(a,b,c,d)}(x)$. Given a permutation $\sigma = \sigma_1\sigma_2 \ldots \sigma_n \in \mathcal{S}_n$, we let the reverse of σ, σ^r, be defined by $\sigma^r = \sigma_n \ldots \sigma_2\sigma_1$, and the complement of σ, σ^c, be defined by $\sigma^c = (n+1-\sigma_1)(n+1-\sigma_2)\ldots(n+1-\sigma_n)$. It is easy to see that the family of generating functions $Q_{\tau^r}^{(a,b,c,d)}(t,x)$, $Q_{\tau^c}^{(a,b,c,d)}(t,x)$, and $Q_{(\tau^r)^c}^{(a,b,c,d)}(t,x)$ can be obtained from the family of generating functions $Q_{\tau}^{(a,b,c,d)}(t,x)$.

2 Results on generating functions $Q_{123}^{(a,b,c,d)}(t,x)$ and their coefficients

In this section, we shall list a sample of results about the generating functions $Q_{123}^{(a,b,c,d)}(t,x)$ and the coefficients $Q_{123}^{(a,b,c,d)}(t,x)\big|_{t^n x^k}$.

It is easy to see that $\mathcal{S}_n(123)$ is closed under the operation reverse-complement. Thus we have the following lemma.

Lemma 1. *For any $a, b, c, d \in \{\emptyset\} \cup \mathbb{N}$, $Q_{n,123}^{(a,b,c,d)}(x) = Q_{n,123}^{(c,d,a,b)}(x)$.*

Next it is obvious that if there is a σ_i in $\sigma = \sigma_1 \ldots \sigma_n \in \mathcal{S}_n$ such that σ_i matches $\mathrm{MMP}(a,b,c,d)$ where $a, c \geq 1$, then σ contains an occurrence of 123. Thus there are no permutations $\sigma \in \mathcal{S}_n(123)$ that can match a quadrant marked mesh pattern $\mathrm{MMP}(a,b,c,d)$ where $a, c \geq 1$. Thus if $a \geq 1$, then $Q_{123}^{(a,b,0,d)}(t,x) = Q_{123}^{(a,b,\emptyset,d)}(t,x)$.

Given an $n \times n$ square, we will label the coordinates of the columns from left to right with $0, 1, \ldots, n$ and the coordinates of the rows from top to bottom with $0, 1 \ldots, n$. A Dyck path is a path made up of unit down-steps D and unit right-steps R which starts at $(0,0)$ and ends at (n,n) which stays on or below the diagonal $x = y$. Many of our results are proved using a bijection of Krattenthaler [7] $\Phi : S_n(132) \to D_n$ and a bijection of Elizalde and Deutsch [2] $\Psi : \mathcal{S}_n(123) \to \mathcal{D}_n$.

Given any permutation $\sigma = \sigma_1 \ldots \sigma_n \in \mathcal{S}_n(132)$, we write it on an $n \times n$ table by placing σ_i in the i^{th} column and σ_i^{th} row, reading from bottom to top. Then, we shade the cells to the north-east of the cell that contains σ_i. Then the path $\Phi(\sigma)$ is the path that goes along the south-west boundary of the shaded cells. For example, this process is pictured on the left in Figure 2 in this case where $\sigma = 867943251 \in \mathcal{S}_9(132)$. Given $\sigma = \sigma_1 \ldots \sigma_n$, we say that σ_j a left-to-right mininum of σ if $\sigma_i > \sigma_j$ for all $i < j$. It is easy to see that the left-to-right minima of σ correspond to peaks of the path $\Phi(\sigma)$, i.e., they occupy cells along the inside boundary of the $\Phi(\sigma)$ that correspond to a down step D immediately followed by a right-step R. For this reason, we shall often refer to the left-to-right minimum as *peaks* of the σ and the remaining elements in σ as *non-peaks*.

The map Φ^{-1} is easy to describe. That is, given a Dyck path P, we first mark every cell corresponding to a peak of the path with a $\times$. Then we look at the rows and columns which do not have a cross. Starting form the left-most column, that does not contain a cross, we put a cross in the lowest possible row without a cross that lies above the path.

The map $\Psi : S_n(123) \to D_n$ is defined by the exact same process. For example, $\Psi(869743251)$ is pictured on the right in Figure 2. The map Φ^{-1} is also easy to describe. That is, given a Dyck path P, we first mark every cell corresponding to a peak of the path with a $\times$. Then we look at the rows and columns which do not have a cross. Starting form the left-most column, that does not contain a cross, we put a cross in the highest possible row without a cross that lies above the path. The map $\Psi^{-1} \circ \Phi$ give a bijection between $S_n(132)$ and $S_n(123)$ which is pictured in Figure 2. This bijection allows us to prove the following theorem.

Theorem 1. *For any $k > 0$ and $\ell, m \geq 0$, $Q_{123}^{(k,\ell,0,m)}(t,x) = Q_{123}^{(k,\ell,\emptyset,m)}(t,x) = Q_{132}^{(k,\ell,\emptyset,m)}(t,x)$.*

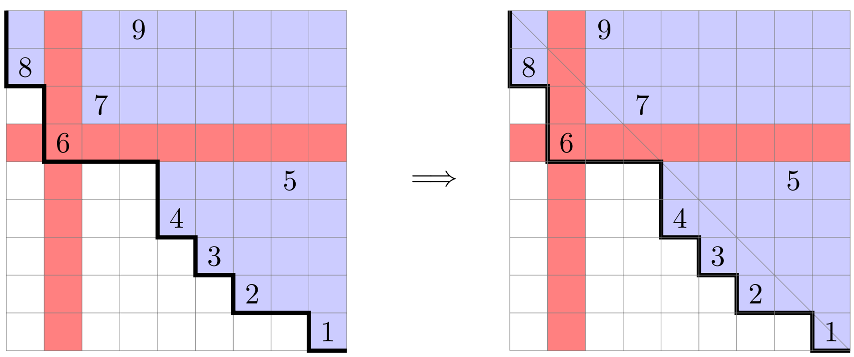

Figure 2: $\mathcal{S}_n(132)$ to $\mathcal{S}_n(123)$ keeps MMP$(k, \ell, \emptyset, m)$

Thus to compute the generating functions $Q_{123}^{k,\ell,0,m}(x,t)$ where $k > 0$, we need only compute the generating functions $Q_{132}^{k,\ell,\emptyset,m}(x,t)$ which can easily be computed using the techniques Kiteav, Remmel, and Tiefenbruck in [4, 5, 6]. In fact, the only generating functions of the form $Q_{132}^{k,\ell,\emptyset,m}(x,t)$ computed in [4, 5, 6] are $Q_{132}^{k,0,\emptyset,0}(x,t)$ where $k \geq 0$. We can find the generating function $Q_{132}^{k,\ell,\emptyset,m}(x,t)$ for any $k, \ell, m \geq 0$. We list a couple of examples of our results below.

Theorem 2.

$$Q_{132}^{(0,0,\emptyset,0)}(t,x) = \frac{1+t-tx-\sqrt{(1+t-tx)^2-4t}}{2t}.$$

For $k > 0$,

$$Q_{132}^{(0,k,\emptyset,0)}(t,x) = \frac{1+t\sum_{i=1}^{k-1} C_{i-1}t^{i-1}(Q_{132}^{(0,k-i,\emptyset,0)}(t,x) - Q_{132}^{(0,0,\emptyset,0)}(t,x))}{1-tQ_{132}^{(0,0,\emptyset,0)}(t,x))}.$$

Theorem 3. *For all $k, \ell > 0$,*

$$Q_{132}^{(k,\ell,\emptyset,0)}(t,x) = 1 + t\sum_{i=1}^{\ell-1} C_{i-1}t^{i-1}Q_{132}^{(k,\ell-i,\emptyset,0)}(t,x) + (Q_{132}^{(k-1,\ell,\emptyset,0)}(t,x) - \sum_{i=0}^{\ell-2} C_i t^i)Q_{132}^{(k,0,\emptyset,0)}(t,x).$$

Theorem 1 implies that the coefficients of x^k in polynomials of the form $Q_{n,123}^{(a,b,0,d)}(x)$ and $Q_{n,123}^{(0,d,a,b)}(x)$ can be found from the coefficients of x^k in polynomials of the form $Q_{n,132}^{(a,b,\emptyset,d)}(x)$. We have the following theorem about the coefficients of x^0 and x^1 in functions $Q_{n,132}^{(a,b,\emptyset,d)}(x)$.

Theorem 4. *$Q_{n,132}^{(k,\ell,\emptyset,m)}(x)\big|_{x^0} = Q_{n,132}^{(k,\ell,0,m)}(x)\big|_{x^0}$ and $Q_{n,132}^{(k,\ell,\emptyset,m)}(x)\big|_{x^1} = Q_{n,132}^{(k,\ell,0,m)}(x)\big|_{x^1}$.*

The reason that this theorem is interesting is that in [4, 5, 6] many explicit formulas were developed for the coefficients $Q_{n,132}^{(k,\ell,0,m)}(x)\big|_{x^0}$ and $Q_{n,132}^{(k,\ell,0,m)}(x)\big|_{x^1}$.

We also prove the following theorems which give formula about the highest power of x in all generating functions $Q_{n,132}^{(a,b,c,d)}(x)$.

Theorem 5. *If $n \geq k+\ell+1$, then*

$$\begin{aligned}
Q_{n,123}^{(0,k,0,\ell)}(x)\big|_{x^{n-k-\ell}} &= Q_{n,132}^{(0,k,0,\ell)}(x)\big|_{x^{n-k-\ell}} = C_k C_{n-k-\ell} C_\ell, \\
Q_{n,123}^{(\emptyset,k,\emptyset,\ell)}(x)\big|_{x^{n-k-\ell}} &= Q_{n,132}^{(\emptyset,k,\emptyset,\ell)}(x)\big|_{x^{n-k-\ell}} = C_k C_\ell, \\
Q_{n,123}^{(0,k,\emptyset,\ell)}(x)\big|_{x^{n-k-\ell}} &= Q_{n,132}^{(0,k,\emptyset,\ell)}(x)\big|_{x^{n-k-\ell}} = C_k C_\ell, \text{ and} \\
Q_{n,123}^{(k,\ell,\emptyset,0)}(x)\big|_{x^{n-k-\ell}} &= Q_{n,132}^{(k,\ell,\emptyset,0)}(x)\big|_{x^{n-k-\ell}} = \frac{k+1}{k+\ell+1}\binom{k+2\ell}{\ell}.
\end{aligned}$$

Theorem 6. *For $n \geq k+\ell+m+1$ and $k>0$,*

$$Q_{n,123}^{(k,\ell,\emptyset,m)}(x)\big|_{x^{n-k-\ell-m}} = Q_{n,132}^{(k,\ell,\emptyset,m)}(x)\big|_{x^{n-k-\ell-m}} = \frac{(k+1)^2}{(k+\ell+1)(k+m+1)}\binom{k+2\ell}{\ell}\binom{k+2m}{m}.$$

By Lemma 1, $Q_{123}^{(a,b,0,d)}(t,x) = Q_{123}^{(0,d,a,b)}(t,x)$ so that the only generating functions $Q_{123}^{(a,b,c,d)}(t,x)$ which we can not compute via Theorem 1 are generating functions of the form $Q_{123}^{(0,b,0,d)}(t,x)$. To compute generating functions of the form $Q_{123}^{(0,b,0,d)}(t,x)$, we must use other methods.

We start by considering generating functions of the form $Q_{123}^{(0,k,0,0)}(t,x)$. In this case, it will be useful to separately track peaks and non-peaks. Thus if $\sigma = \sigma_1 \ldots \sigma_n \in \mathcal{S}_n(123)$, then we will say that σ_i matches the pattern $\mathrm{MMP}(0,\binom{k_1}{k_2},0,0)$ if σ_i is peak of σ and it matches the pattern $\mathrm{MMP}(0,k_1,0,0)$ or σ_i is a non-peak of σ and it matches the pattern $\mathrm{MMP}(0,k_2,0,0)$. Then we define

$$Q_{123}^{(0,\binom{k_1}{k_2},0,0)}(t,x_0,x_1) = \sum_{n=0}^{\infty} t^n Q_{n,123}^{(0,\binom{k_1}{k_2},0,0)}(x_0,x_1)$$

where $Q_{n,123}^{(0,\binom{k_1}{k_2},0,0)}(x_0,x_1) = \sum_{\sigma\in\mathcal{S}_n(123)} x_0^{\#\text{ MMP}(0,k_1,0,0)\text{-mch of peaks}} x_1^{\#\text{ MMP}(0,k_2,0,0)\text{-mch of non-peaks}}$.

We can show that $Q_{n,123}^{(0,\binom{k_1}{k_2},0,0)}(x_0,x_1)$ the polynomials satisfy simple recursions which lead to recursive formulas to compute $Q_{123}^{(0,\binom{k_1}{k_2},0,0)}(t,x_0,x_1)$. For example, we prove that

Theorem 7. *For all $k_1,k_2>0$, we have*

$$\begin{aligned}
Q_{123}^{(0,\binom{k_1}{0},0,0)}(t,x_0,x_1) &= \frac{1}{1-tx_1 Q_{123}^{(0,\binom{0}{0},0,0)}(t,x_0,x_1)}\Bigg(1+tQ_{123}^{(0,\binom{k_1-1}{0},0,0)}(t,x_0,x_1) \\
&\quad + tx_1\sum_{i=2}^{k_1-1} t^{i-1} Q_{i-1,123}^{(0,\binom{0}{0},0,0)}(1,x_1) Q_{123}^{(0,\binom{k_1-i}{0},0,0)}(t,x_0,x_1) \\
&\quad - tx_1 Q_{123}^{(0,\binom{0}{0},0,0)}(t,x_0,x_1)\sum_{i=0}^{k_1-2} t^i Q_{i-1,123}^{(0,\binom{0}{0},0,0)}(1,x_1)\Bigg).
\end{aligned}$$

For generating functions $Q_{123}^{(0,k,0,\ell)}(x,t)$, we divide the graph of a permutation into several regions and have the following theorem to enumerate the coefficients $Q_{n,123}^{(0,k,0,\ell)}(x)\big|_{x^s}$.

Theorem 8. *For any* 123*-avoiding permutation* $\sigma = \sigma_1 \ldots \sigma_n$*,* σ_j *matches* $\mathrm{MMP}(0,k,0,\ell)$ *in* σ *if and only if, in the graph* $G(\sigma)$ *of* σ*,* (j,σ_j) *does not lie in the top* k *rows or the bottom* ℓ *rows and it does not lie in the left-most* k *columns or the right-most* ℓ *columns. Thus*

$$\mathrm{mmp}^{(0,k,0,\ell)}(\sigma) = \Big|\{j|k < j \leq n-\ell \text{ and } k < \sigma_j \leq n-\ell\}\Big|.$$

Using Theorem 8, we can calculate $Q_{123}^{(0,k,0,\ell)}(t,x)$ for k,l not too big. For example,

Theorem 9. *For* $n \geq 4$*,* $Q_{123}^{(0,1,0,1)}(t,x)\big|_{t^nx^k} = 0$ *unless* $k \in \{n-4, n-3, n-2\}$ *and*

$$\begin{aligned}
Q_{123}^{(0,1,0,1)}(t,x)\big|_{t^nx^{n-4}} &= C_n - 2C_{n-1} + C_{n-2} - 2, \\
Q_{123}^{(0,1,0,1)}(t,x)\big|_{t^nx^{n-3}} &= 2C_{n-1} - 2C_{n-2} + 2, \text{ and} \\
Q_{123}^{(0,1,0,1)}(t,x)\big|_{t^nx^{n-2}} &= C_{n-2}
\end{aligned}$$

where C_n *denote the* n^{th} *Catalan number.*

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