# Stirling Numbers & Euler's Finite Difference Theorem

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Generalizing from an interesting Laplace transform identity, Katsuura [2] proves, via induction, that for $k, n \in \mathbb{Z}$ with $0 \leq k \leq n$,

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} (jx+y)^k = \begin{cases} (-x)^n n!, & k = n, \text{ and} \\ 0, & 0 \leq k < n \end{cases} \tag{0.1}$$

for any $x, y \in \mathbb{C}$. At first glance it seems odd that the righthand side of this equation never contains $y$; however Equation (0.1) is a special case of the more general polynomial identity given in Equation (0.2). This equation involves Stirling numbers of the second kind, denoted by $\left\{ {n \atop k} \right\}$, and we see that powers of $y$ only occur if $k > n$:

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} (jx+y)^k = (-1)^n n! \sum_{i=0}^{k} \binom{k}{i} \left\{ {i \atop n} \right\} x^i y^{k-i}. \tag{0.2}$$

Equation (0.2) may be proved combinatorially via involutions on colored words, but it turns out, unsurprisingly, that the formula itself is not totally new. Indeed, it is a consequence of Euler's Finite Difference Theorem [3, pg. 68].

**Theorem 1** (Euler's Finite Difference Theorem)**.** *Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$ be a complex polynomial and let $n$ be a nonnegative integer. Then*

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} f(j) = \begin{cases} (-1)^n a_n n!, & k = n, \text{ and} \\ 0, & 0 \leq k < n. \end{cases}$$

Our goal is to generalize Theorem 1 and then use that generalization for our own combinatorial purposes. To that end, we start with the following well-known formula involving $\left\{ {n \atop k} \right\}$:

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} j^k = (-1)^n n! \left\{ {k \atop n} \right\}.$$

Letting $g(x) = a_0 + a_1 x + a_2 x^2 + \cdots$, we multiply both sides of the preceding equation by $a_k$ and sum over all $k \geq 0$ to obtain our desired generalization of Theorem 1:

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} g(j) = (-1)^n n! \sum_{k \geq 0} a_k \left\{ {k \atop n} \right\}. \tag{0.3}$$

By substituting in for $g(j)$, Equation (0.3) yields a vast assortment of combinatorial identities. For instance, letting $g(j) = (jx+y)^k$ yields Equation (0.2), which proves our motivating example, whereas letting $g(j) = 1$ yields the well-known alternating binomial sum identity

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} = 0.$$

Another well-known identity that follows immediately from the proper choice of $g(j)$ is the orthogonality relation between $\left\{ {n \atop k} \right\}$ and the signed Stirling numbers of the first kind, denoted by $\left[ {n \atop k} \right]$:

$$\sum_{k\geq 0} \left[ {m \atop k} \right]\left\{ {k \atop n} \right\} \quad = \quad \begin{cases} 1, & k = n, \text{ and} \\ 0, & 0 \leq m < n. \end{cases}$$

For another application, we go back a few decades to Broder's $r$-Stirling numbers of the second kind [1], denoted by $\left\{ {n \atop k} \right\}_r$ and defined as the number of set partitions of $\{1, 2, \ldots, n\}$ into $k$ nonempty, unordered parts such that $1, 2, \ldots, r$ are in distinct parts. Rewriting Equation (32) from [1], we get that

$$\left\{ {k+r \atop n+r} \right\}_r \quad = \quad \sum_{i=n}^{k} \binom{k}{i} \left\{ {i \atop n} \right\} r^{k-i}, \tag{0.4}$$

and letting $g(j) = (j+r)^k$ in Equation (0.3), we obtain, via Equation (0.4), the following result.

**Theorem 2.** *Let $k, n, r$ be nonnegative integers with $r \geq 1$. Then*

$$(-1)^n n! \left\{ {k+r \atop n+r} \right\}_r \quad = \quad \sum_{j=0}^{n} (-1)^j \binom{n}{j} (j+r)^k.$$

Theorem 2 then gives a new way to find the exponential generating function of $\left\{ {k+r \atop n+r} \right\}_r$:

$$\sum_{k\geq 0} \left\{ {k+r \atop n+r} \right\}_r \frac{z^k}{k!} \quad = \quad \frac{e^{zr}}{n!}(e^z - 1)^n.$$

## References


[1] A. Z. Broder, The $r$-Stirling numbers, *Discrete Mathematics* **49** (1984), 241–259

[2] H. Katsuura, Summations involving binomial coefficients, *The College Mathematics Journal*, **40** (2009), no. 4, 275–278.

[3] J. Quaintance & H. W. Gould, *Combinatorial Identities for Stirling Numbers*, World Scientific Publishing, 1st edition, 2015